

# On the uncertainty relations and quantum measurements: conventionalities, shortcomings, reconsiderations

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## Abstract

Unsolved controversies about uncertainty relations and quantum measurements still persists nowadays. They originate around the shortcomings regarding the conventional interpretation of uncertainty relations. Here we show that the respective shortcomings disclose veridic and unavoidable facts which require the abandonment of the mentioned interpretation. So the primitive uncertainty relations appear as being either thought fictions or fluctuations formulae. Subsequently we reveal that the conventional approaches of quantum measurements are grounded on incorrect premises. We propose a new approach in which : (i) the quantum observables are considered as generalized stochastic variables, (ii) the view is focused only on the pre-existent state of the measured system, without any interest for the collapse of the respective state, (iii) a measurement is described as an input-output transformation which modify the probability density and current but preserve the expressions of the operators. The measuring uncertainties are evaluated as changes in the probabilistic estimators of observables. Related to different observables we do not find reasons of principle neither for uncertainties-connections nor for measuring compatibility/incompatibility.

*Keywords* Uncertainty relations, conventional interpretation, shortcomings, quantum measurements, reconsiderations.

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## 1 Introduction

Debates about the uncertainty relations (UR) are present in a large number of early as well as recent publications (for a significant bibliography see [1–7]). In a direct or dissimulated manner most of the respective debates are connected with the so-called conventional interpretation of UR (CIUR), promoted by the Copenhagen School and its partisans. Often CIUR is mentioned as fragmentary

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excerpts from a diffuse collection of statements. For all that it can be confined around a restricted number of basic ideas (see below the next section). Less mentioned is the fact that CIUR ideas are troubled by a number of still unsolved shortcomings. As a rule, in the main stream of recent publications, the alluded shortcomings are underestimated (through unnatural solutions or even by omission).

The most known shortcomings of CIUR regards [1,8–24] the following pairs of canonically conjugated observables  $L_z - \varphi$ ,  $N - \phi$  and  $E - t$  ( $L_z = z$  component of angular momentum,  $\varphi =$  azimuthal angle,  $N =$  number,  $\phi =$  phase,  $E =$  energy,  $t =$  time). The respective pairs show anomalies in respect with the usual (Robertson - Schrodinger) version of theoretical UR and, consequently, they are in conflict with CIUR ideas. For solving the conflict in literature the current attitude is to preserve CIUR as an immovable doctrine and to adjust adequately the expressions of theoretical UR (for details see the Sections 3 and 4). But as an intriguing fact the mentioned adjustments differ among them both quantitatively and qualitatively. So, until now, an agreement regarding the mentioned conflict does not exist in the scientific literature.

In the alluded circumstances we think that an investigation ab origine of the facts is of major interest. Such an investigation has to include:

- (i) firstly, a search of the primary mathematical source regarding the theoretical UR and
- (ii) secondly, a consideration of the respective source as a standard (reference element) in the appreciation of things.

An investigation of the mentioned kind we propose in Section 5. Thus we find that the searched source/standard is a Cauchy-Schwarz relation that must replace the usual UR in debates regarding the above mentioned pairs of observables. But the respective finding disclose insurmountable contradictions with CIUR.

A good appraisal of the proposed investigation requires to examine also other things which reveal additional shortcomings of CIUR. Some of the respective things were mentioned in publications only occasionally and without noticeable impact in scientific community. But a significant class of such things can be collected by a careful exploration of the corresponding facts and literature (see Section 6). A minute examination of the elements of the respective class shows that each of them contradict in an indubitable manner one or more basic ideas (**BI**) on which CIUR relies. Moreover the ensemble of the respective contradictions, together with the inherent anomalies shown by the above precised pairs of conjugated observables, incriminate in an insurmountable way the whole set of the mentioned BI. So CIUR appears in a situation of indubitable failure and, consequently, it must be re-evaluated and abandoned as an incorrect doctrine (for details see Section 7).

The mentioned re-evaluation of CIUR entails a complete disconnection of UR from their supposed significance regarding the quantum measurements (QMS). The description of QMS is a question approached in a large number of controversial debates (see publications [1–6,25–29] and references). Most of the respective debates promoted the presumption that the alluded description must

be included within the framework of quantum mechanics (QM). Now we note that, in spite of the mentioned disconnection of UR, the description of QMS must remain a natural subject for scientific investigations. On the other hand we think that the above alluded old presumption is unjustified from the viewpoint of real physics practice . We opine that, according to the respective practice, the description of QMS must be considered as a distinct and additional task in respect with the purposes of usual QM. Our opinions regarding the reconsideration of QMS description are argued, presented and exemplified in Sections 8-11 and in Annex A.

## 2 Basic ideas of CIUR

In the main CIUR concerns on the purpose to give a unique and generic interpretation for the thought-experimental (*te*) relation

$$\Delta_{te}A \cdot \Delta_{te}B \geq \hbar \quad (1)$$

and for the theoretical formula

$$\Delta_{\psi}A \cdot \Delta_{\psi}B \geq \frac{1}{2} \left| \left\langle \left[ \hat{A}, \hat{B} \right] \right\rangle_{\psi} \right| \quad (2)$$

Relation (1) regards the *te*-uncertainties  $\Delta_{te}A$  and  $\Delta_{te}B$  for conjugated observables  $A$  and  $B$  (like  $A = x =$  coordinate and  $B = p =$  momentum ). It was introduced [30,31] by means of so-called thought (or mental) experiments. Formula (2) (known as Robertson - Schrödinger UR) derives from the mathematical formalism of QM (for some details see also Section 5).

Motivated by the mentioned concern CIUR was widely popularized and, explicitly or implicitly, it is agreed in a large number of old as well as recent publications. But as a strange aspect, in dissonance with such facts, in its partisan literature CIUR is often presented so vaguely and fragmentarily that it seems to be rather an intricate conception but not a well defined doctrine. However, in spite of the respective aspect, one can see [32–34] that in fact CIUR is founded on a restricted number of basic ideas (**BI**). Here we report the respective **BI** as follows:

**BI.1:** *Quantities  $\Delta_{te}A$  and  $\Delta_{\psi}A$  from relations (1) and (2), denoted by a unique symbol  $\Delta A$ , have similar significance of measuring uncertainty for the observable  $A$ . Consequently the respective relations have the same generic interpretation as UR regarding the simultaneous measurements of observables  $A$  and  $B$ .*

**BI.2:** *In case of a solitary observable  $A$  the quantity  $\Delta A$  always can have an unbounded small value. Therefore such an observable can be measured without uncertainty in all cases of systems and states.*

**BI.3:** *When two observables  $A$  and  $B$  are commutable (i.e.  $[\hat{A}, \hat{B}] = 0$ ) relation (2) allows for the quantities  $\Delta A$  and  $\Delta B$  to be unlimitedly small at the same time. That is why such observables can be measured simultaneously and*

without uncertainties for any system or state. Therefore they are considered as compatible.

**BI.4:** If two observables  $A$  and  $B$  are noncommutable (i.e.  $[\hat{A}, \hat{B}] \neq 0$ ) relation (2) shows that the quantities  $\Delta A$  and  $\Delta B$  can be never reduced concomitantly to null values. For that reason such observables can be measured simultaneously only with non-null and interconnected uncertainties, irrespective of the system or state. Consequently such observables are considered as incompatible.

**BI.5:** Relations (1) and (2), Planck's constant  $\hbar$  as well as the measuring peculiarities noted in **BI.4** are typically QM things which have not analogy in classical (non-quantum) macroscopic physics .

### 3 Cases of angular observables

In its integrity (previously delimited through **BI.1-5**) CIUR is vulnerable to shortcomings connected with the pairs of angular observables  $L_z - \varphi$  and  $N - \phi$  (z-component of angular momentum - azimuthal angle, respectively number - phase). Some of the respective shortcomings were debated in publications from the last decades (see [1, 8-18] and references).

As regards the  $L_z - \varphi$  pair the mentioned debates revealed the following facts: According to the usual procedures of QM,  $L_z$  and  $\varphi$  should be described by the conjugated operators

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \varphi}, \quad \hat{\varphi} = \varphi. \quad (3)$$

respectively by the commutation relation

$$[\hat{L}_z, \hat{\varphi}] = -i\hbar \quad (4)$$

So for the alluded pair the CIUR's basic formula (2) requires directly the relation

$$\Delta_\psi L_z \cdot \Delta_\psi \varphi \geq \frac{\hbar}{2} \quad (5)$$

On the other hand the angular states of some systems (which will be specified below) are described by the wave functions

$$\psi_m(\varphi) = (2\pi)^{-\frac{1}{2}} e^{im\varphi} \quad (6)$$

with  $\varphi \in [0, 2\pi)$  and  $m = 0, \pm 1, \pm 2, \dots$  For such states one obtains

$$\Delta_\psi L_z = 0, \quad \Delta_\psi \varphi = \frac{\pi}{\sqrt{3}} \quad (7)$$

But these expressions are incompatible with relation (5).

For avoiding the mentioned incompatibility many publications promoted the conception that in the case of  $L_z - \varphi$  pair the usual procedures of QM do not

work correctly. Consequently it was accredited the idea that formula (5) must be prohibited and replaced by adjusted  $\Delta_\psi L_z - \Delta_\psi \varphi$  relations resembling with (2). So, along the years, a lot of such adjusted relations were proposed. In the main the respective relations are expressed in one of the following forms:

$$\Delta_\psi L_z \cdot \Delta_\psi f(\varphi) \geq \hbar \left| \langle g(\varphi) \rangle_\psi \right| \quad (8)$$

$$(\Delta_\psi L_z)^2 + \hbar^2 (\Delta_\psi u(\varphi))^2 \geq \hbar^2 \langle v(\varphi) \rangle_\psi^2 \quad (9)$$

$$\Delta_\psi L_z \cdot \Delta_\psi \varphi \geq \frac{\hbar}{2} |1 - 2\pi |\psi(2\pi - 0)|| \quad (10)$$

where  $\psi(2\pi - 0) := \lim_{\varphi \rightarrow 2\pi - 0} \psi(\varphi)$ .

In (8) - (9)  $f(\varphi)$ ,  $g(\varphi)$ ,  $u(\varphi)$  and  $v(\varphi)$  denote various adjusting functions of  $\varphi$  asserted by means of some circumstantial (and more or less fictitious) considerations.

A minute examination of the facts shows that, in essence, the relations (8) - (10) are troubled by shortcomings revealed within the following remarks (**R**).

**R.1:** None of the respective relations is agreed unanimously as a correct  $\Delta_\psi L_z - \Delta_\psi \varphi$  relation able to replace formula (5).

**R.2:** Mathematically the mentioned relations are not mutually equivalent.

**R.3:** Relations (8) - (9) have no rational justifications in the usual formalism of QM (that however works very well in a huge number of applications).

**R.4:** The relation (10) is correct from the usual QM perspective (see formula (41) in Section 5) but it conflicts with the CIUR's idea **BI.4** in the case of states described by the wave functions (6).

In respect with the  $N - \phi$  pair the situation is as follows. The respective pair refers to a quantum oscillator and it is described by the operators  $\hat{N}$  and  $\hat{\phi}$  introduced by relations

$$\hat{a} = e^{i\hat{\phi}} \sqrt{\hat{N}}, \quad \hat{a}^+ = \sqrt{\hat{N}} e^{-i\hat{\phi}} \quad (11)$$

where  $\hat{a}$  and  $\hat{a}^+$  denote the known ladder (lowering and raising) operators. From (11) one finds

$$[\hat{N}, \hat{\phi}] = i \quad (12)$$

CIUR's basic formula (2) requires thus directly the relation

$$\Delta_\psi N \cdot \Delta_\psi \phi \geq \frac{1}{2} \quad (13)$$

On the other hand for an oscillator in an energetic eigenstate one obtains the results

$$\Delta_\psi N = 0, \quad \Delta_\psi \phi = \frac{\pi}{\sqrt{3}} \quad (14)$$

(the last of these results, not mentioned explicitly in many publications, can be obtained easily by means of the wave functions (17) given below). Now one can

see that the results (14) invalidate the relation (13) and so it is revealed the anomaly of  $N - \phi$  pair in respect with CIUR.

Here we also add the next remarks.

**R.5:** Mathematically the situation of the  $N - \phi$  pair is completely similar with that of the  $L_z - \varphi$  pair. The respective similarity can be pointed out as follows. If the wave functions are considered in the  $\phi$ -representation (with  $\phi \in [0, 2\pi)$ ) from (12) results that the operators  $\hat{N}$  and  $\hat{\phi}$  have the expressions

$$\hat{N} = i \frac{\partial}{\partial \phi}, \quad \hat{\phi} = \phi. \quad (15)$$

Then the Schrödinger equation for oscillator take the form

$$\hbar\omega \left( i \frac{\partial}{\partial \phi} + \frac{1}{2} \right) \psi = E\psi \quad (16)$$

where  $E =$  energy and  $\omega =$  angular frequency.

By considering  $\psi(2\pi - 0) = \psi(0)$  and  $E > 0$  it results that in the  $\phi$ -representation a quantum oscillator is described by the wave functions

$$\psi_N(\phi) = (2\pi)^{-\frac{1}{2}} e^{-iN\phi} \quad (17)$$

with  $N = 0, 1, 2, 3, \dots$  (correspondingly  $E = E_N = \hbar\omega(N + 1/2)$ ). Then a direct comparison of pairs of relations (15) and (17) respectively (3) and (6) attests the announced mathematical similarity between the pairs  $N - \phi$  and  $L_z - \varphi$ .

**R.6:** The similarity between the two pairs is also evidenced (perhaps less visible) by the various  $\Delta_\psi N - \Delta_\psi \phi$  adjusted formulae proposed in literature (see [11, 13, 14, 16]) in order to replace (13) and to avoid the  $N - \phi$  anomaly in respect with CIUR. In their essence the respective formulae are completely analogous with the relations (8) - (10) for the  $L_z - \varphi$  pair. Moreover, it is easy to see that the mentioned formulae are troubled by shortcomings which are similar with the ones mentioned above in **R.1 - 4**.

Now let us note that in the  $L_z - \varphi$  case the states described by (6) regards exclusively the restricted class of sharp circular rotations (SRC) around the z-axis. Such SRC are specific for a particle on a circle, for a 1D (or fixed-axis) rotator, and for non-degenerate spherical rotations respectively. One finds examples of systems with spherical rotations in the cases of a particle on a sphere, of 2D or 3D rotators and of an electron in a hydrogen atom respectively. The mentioned rotations are considered as non-degenerate if all the specific (orbital) quantum numbers have well-defined (unique) values.

Here is the place to remind that the situation of the  $L_z - \varphi$  pair must be also discussed in relation with the extended rotations (EXR). By EXR we refer to the quantum torsion pendulum (QTP) and to the degenerate spherical rotations (of the above mentioned systems) respectively. A rotation (motion) is degenerate if the energy of the system is well-specified while the non-energetic quantum numbers (here of orbital nature) take all permitted values.

From the class of EXR let us firstly refer to the case of a QTP which [33,34] can be regarded as a simple QM oscillator. Indeed a QTP which oscillate around the z-axis is characterized by the Hamiltonian

$$\hat{H} = \frac{1}{2I}\hat{L}_z^2 + \frac{1}{2}I\omega^2\varphi^2 \quad (18)$$

Here  $\varphi$  denotes the azimuthal angle with  $\varphi \in (-\infty, \infty)$ ,  $\hat{L}_z$  is the z-component of angular momentum operator defined by (3),  $I$  is the momentum of inertia and  $\omega$  represents the (angular) frequency of torsion oscillation. Then the eigenstates of QTP have energies  $E_N = \hbar\omega(N + 1/2)$  and are described by the wave functions

$$\psi_N(\varphi) = \psi_N(\xi) \propto \exp\left(-\frac{\xi^2}{2}\right) \mathcal{H}_N(\xi), \quad \xi = \varphi\sqrt{\frac{I\omega}{\hbar}} \quad (19)$$

where  $N = 0, 1, 2, 3, \dots$  signifies the oscillation quantum number and  $\mathcal{H}_N(\xi)$  stand for Hermite polynomials of  $\xi$ . In the states (19) for the observables  $L_z$  and  $\varphi$  associated with the operators (3) one obtains

$$\Delta_\psi L_z = \sqrt{\hbar I \omega \left(N + \frac{1}{2}\right)}, \quad \Delta_\psi \varphi = \sqrt{\frac{\hbar}{I \omega} \left(N + \frac{1}{2}\right)} \quad (20)$$

With these expressions for  $\Delta_\psi L_z$  and  $\Delta_\psi \varphi$  one finds that for the considered QTP the  $L_z - \varphi$  pair satisfies the prohibited formula (5).

From the same class of EXR let us now refer to a degenerate state of a particle on a sphere or of a 2D rotator. In such a state the energy is  $E = \hbar^2 l(l+1)/2I$  where the orbital number  $l$  has a well-defined value ( $I =$  moment of inertia). In the same state the magnetic number  $m$  can take all the values  $-l, -l+1, \dots, -1, 0, 1, \dots, l-1, l$ . Then the mentioned state is described by a wave function of the form

$$\psi_l(\theta, \varphi) = \sum_{m=-l}^l c_m Y_{lm}(\theta, \varphi) \quad (21)$$

Here  $\theta$  and  $\varphi$  denote the polar and azimuthal angle respectively ( $\theta \in [0, \pi]$ ,  $\varphi \in [0, 2\pi)$ ),  $Y_{lm}(\theta, \varphi)$  are the spherical functions and  $c_m$  represent complex coefficients which satisfy the normalization condition  $\sum_{m=-l}^l |c_m|^2 = 1$ . With the expressions (3) for the operators  $\hat{L}_z$  and  $\hat{\varphi}$  in a state described by (21) one obtains

$$(\Delta_\psi L_z)^2 = \sum_{m=-l}^l |c_m|^2 \hbar^2 m^2 - \left[ \sum_{m=-l}^l |c_m|^2 \hbar m \right]^2 \quad (22)$$

$$\begin{aligned}
(\Delta_\psi \varphi)^2 = & \sum_{m=-l}^l \sum_{k=-l}^l c_m^* c_k (Y_{lm}, \varphi^2 Y_{lk}) - \\
& - \left[ \sum_{m=-l}^l \sum_{r=-l}^l c_m^* c_r (Y_{lm}, \varphi Y_{lr}) \right]^2
\end{aligned} \tag{23}$$

where  $(f, g)$  denotes the scalar product of the functions  $f$  and  $g$ .

By means of the expressions (22) and (23) one finds that in the case of alluded degenerated EXR, described by (21) it is possible for the prohibited formula (5) to be satisfied. Such a possibility is conditioned by the concrete values of the  $c_m$  coefficients.

**R.7:** The facts presented above in this section prove that, in reality, CIUR is unable to give a natural and unitary approach of the problems connected with the pairs of angular observables  $L_z - \varphi$  and  $N - \phi$ . The respective unability can not be remedied in a way by means of the inner resources of CIUR. Consequently it must be recorded as a major and unavoidable shortcoming of CIUR.

## 4 The case of energy and time

Another pair of (canonically) conjugated observables which are unconformable in relation with the CIUR ideas is given by energy  $E$  and time  $t$ . That is why the respective pair was the subject of a large number of (old as well as recent) controversial discussions (see [20–24] and references). The alluded discussions were generated by the following observations. On one hand  $E$  and  $t$ , as conjugated observables have to be described in terms of QM by the operators

$$\hat{E} = i\hbar \frac{\partial}{\partial t}, \quad \hat{t} = t. \tag{24}$$

respectively by the commutation relation

$$[\hat{E}, \hat{t}] = i\hbar \tag{25}$$

In accordance with (2) such description require the relation

$$\Delta_\psi E \cdot \Delta_\psi t \geq \frac{\hbar}{2} \tag{26}$$

On the other hand because in usual QM the time  $t$  is a deterministic but not a stochastic variable for any quantum situation (system and state) one finds the expressions

$$\Delta_\psi E = \text{a finite quantity}, \quad \Delta_\psi t \equiv 0 \tag{27}$$

But these expressions invalidate the relation (26) and consequently show an anomaly in respect with the CIUR ideas (especially with **BI.4**). For avoiding



the alluded anomaly CIUR partisans invented a lot of adjusted  $\Delta E - \Delta t$  formulae destined to substitute the questionable relation (26)(see [1, 20–24] and references). The mentioned formulae can be written in the generic form

$$\Delta_v E \cdot \Delta_v t \geq \frac{\hbar}{2} \quad (28)$$

Here  $\Delta_v E$  and  $\Delta_v t$  have various ( $v$ ) significances such as: (i)  $\Delta_1 E =$  line-breadth of the spectrum characterizing the decay of an excited state and  $\Delta_1 t =$  half-life of the respective state, (ii)  $\Delta_2 E = \hbar \Delta \omega =$  spectral width (in terms of frequency  $\omega$ ) of a wave packet and  $\Delta_2 t =$  temporal width of the wave packet, (iii)  $\Delta_3 E = \Delta_\psi E$  and  $\Delta_3 t = \Delta_\psi A \cdot \left( d \langle A \rangle_\psi / dt \right)^{-1}$ , with  $A =$  an arbitrary observable.

Note that in spite of the efforts and imagination implied in the disputes connected with the formulae (28) the following remarks remain of topical interest.

**R.8:** The diverse formulae from the family (28) are not mutually equivalent from a mathematical viewpoint. Moreover they have no natural justification in the framework of usual QM (that however give a huge number of good results in applications).

**R.9:** In the specific literature (see [1, 20–24] and references) none of the formulae (28) is agreed unanimously as a correct substitute for relation (26).

**R.10:** Consequently the applicability of the CIUR ideas to the  $E - t$  pair persists in our days as a still unsolved question.

## 5 An investigation ab origine of the facts

In its essence the above presented conflict of CIUR with the mentioned pairs of observables regards the applicability of the theoretical formula (2). A clearing up of the facts requires an investigation ab origine of the conditions/range of validity for the respective formula. Such an investigation can be done as follows.

Let us consider a quantum system whose state and observables  $A_j$  ( $j = 1, 2, \dots, r$ ) are described by the wave function  $\psi$  and by the operators  $\hat{A}_j$  respectively. If  $(f, g)$  denote the scalar product of the functions  $f$  and  $g$  the quantity  $\langle A_j \rangle = (\psi, \hat{A}_j \psi)$  represents the mean (expected) value of the observable  $A_j$  in the mentioned state. Because  $A_j$  are stochastic variables they show fluctuations (deviations from the mean values). The respective fluctuations are described (in a first order approximation) by means of the correlations  $C_{jk} = (\delta_\psi \hat{A}_j \psi, \delta_\psi \hat{A}_k \psi)$  where  $\delta_\psi \hat{A}_j = \hat{A}_j - \langle A_j \rangle$ . It is easily to see that the alluded correlations satisfy the following two relations

$$\left( \delta_\psi \hat{A}_j \psi, \delta_\psi \hat{A}_k \psi \right)^* = \left( \delta_\psi \hat{A}_k \psi, \delta_\psi \hat{A}_j \psi \right) \quad (29)$$

$$\left\| \sum_{j=1}^r \lambda_j \delta_\psi \hat{A}_j \psi \right\|^2 = \sum_{j=1}^r \sum_{k=1}^r \lambda_j^* \lambda_k \left( \delta_\psi \hat{A}_j \psi, \delta_\psi \hat{A}_k \psi \right) \geq 0 \quad (30)$$

which imply the notations:  $f^*$  = complex conjugate of  $f$ ,  $\|g\|$  = norm of  $g$ ,  $\lambda_j (j = 1, 2, \dots, r)$  = a set of arbitrary and complex parameters. The relations (29) and (30) show that the set of correlations  $C_{jk}$  constitutes a Hermitian and non-negatively defined matrix. Then in accordance with the matrix algebra [36] can be written the formula

$$\det [C_{jk}] = \det \left[ \left( \delta_\psi \hat{A}_j \psi, \delta_\psi \hat{A}_k \psi \right) \right] \geq 0 \quad (31)$$

where  $\det [C_{jk}]$  denotes the determinant with elements  $C_{jk}$ . For two observables  $A_1 = A$  and  $A_2 = B$  from (31) one obtains

$$\left( \delta_\psi \hat{A} \psi, \delta_\psi \hat{A} \psi \right) \left( \delta_\psi \hat{B} \psi, \delta_\psi \hat{B} \psi \right) \geq \left| \left( \delta_\psi \hat{A} \psi, \delta_\psi \hat{B} \psi \right) \right|^2 \quad (32)$$

i.e. a Cauchy-Schwarz relation for the functions  $\delta_\psi \hat{A} \psi$  and  $\delta_\psi \hat{B} \psi$ . For an observable  $A$  regarded as a stochastic variable the quantity  $\left( \delta_\psi \hat{A} \psi, \delta_\psi \hat{A} \psi \right)^{\frac{1}{2}} = \Delta_\psi A$  represents its standard deviation. From (32) it results directly that the standard deviations  $\Delta_\psi A$  and  $\Delta_\psi B$  of two observables  $A$  and  $B$  satisfy the relation

$$\Delta_\psi A \cdot \Delta_\psi B \geq \left| \left( \delta_\psi \hat{A} \psi, \delta_\psi \hat{B} \psi \right) \right| \quad (33)$$

which can be called *Cauchy-Schwarz formula*. Note that the relations (31) - (33) are always valid (i.e. for all observables, systems and states). The formula (33) implies the less general UR (2) only when the two operators  $\hat{A} = \hat{A}_1$  and  $\hat{B} = \hat{A}_2$  satisfy the conditions

$$\left( \hat{A}_j \psi, \hat{A}_k \psi \right) = \left( \psi, \hat{A}_j \hat{A}_k \psi \right) \quad (j = 1, 2; k = 1, 2) \quad (34)$$

Indeed in such cases one can write the relation

$$\begin{aligned} \left( \delta_\psi \hat{A} \psi, \delta_\psi \hat{B} \psi \right) &= \frac{1}{2} \left( \psi, \left( \delta_\psi \hat{A} \cdot \delta_\psi \hat{B} \psi + \delta_\psi \hat{B} \cdot \delta_\psi \hat{A} \right) \psi \right) - \\ &\quad - \frac{i}{2} \left( \psi, i \left[ \hat{A}, \hat{B} \right] \right) \end{aligned} \quad (35)$$

where the two terms from the right hand side are purely real and imaginary quantities respectively. Therefore in the mentioned cases from (33) one finds

$$\Delta_\psi A \cdot \Delta_\psi B \geq \frac{1}{2} \left| \left\langle \left[ \hat{A}, \hat{B} \right] \right\rangle_\psi \right| \quad (36)$$

i.e. the usual UR (2). The above general framing of UR (2)/(36) suggests that for the here investigated questions it is important to examine the fulfilment of the conditions (34) in each of the considered case. In this sense the following remarks are of direct interest.

**R.11:** In the cases described by the wave functions (6) and (17) for  $L_z - \varphi$  and  $N - \phi$  pairs one finds

$$\left( \hat{L}_z \psi_m, \hat{\varphi} \psi_m \right) = \left( \psi_m, \hat{L}_z \hat{\varphi} \psi_m \right) + i\hbar \quad (37)$$

$$\left( \hat{N}\psi_N(\phi), \hat{\phi}\psi_N(\phi) \right) = \left( \psi_N(\phi), \hat{N}\hat{\phi}\psi_N(\phi) \right) - i \quad (38)$$

**R.12:** For  $L_z - \varphi$  pair in the cases associated with the wave functions (19) and (21) one obtains

$$\left( \hat{L}_z\psi_N(\varphi), \hat{\varphi}\psi_N(\varphi) \right) = \left( \psi_N(\varphi), \hat{L}_z\hat{\varphi}\psi_N(\varphi) \right) \quad (39)$$

$$\begin{aligned} \left( \hat{L}_z\psi_l, \hat{\varphi}\psi_l \right) &= \left( \psi_l, \hat{L}_z\hat{\varphi}\psi_l \right) + \\ &+ i\hbar \left\{ 1 + 2 \operatorname{Im} \left[ \sum_{m=-l}^l \sum_{r=-l}^l c_m^* c_r \hbar m (Y_{lm}, \hat{\varphi} Y_{lr}) \right] \right\} \end{aligned} \quad (40)$$

(where  $\operatorname{Im}[\alpha]$  denotes the imaginary part of  $\alpha$ ).

**R.13:** For any wave function  $\psi(\varphi)$  with  $\varphi \in [0, 2\pi)$  and  $\psi(2\pi - 0) = \psi(0)$  the following relation is generally true

$$\left| \left( \delta_\psi \hat{L}_z \psi, \delta_\psi \hat{\varphi} \psi \right) \right| \geq \frac{\hbar}{2} |1 - 2\pi |\psi(2\pi - 0)|| \quad (41)$$

**R.14:** In the case of energy and time described by the operators (24) for any wave function  $\psi$  one finds

$$\left( \hat{E}\psi, \hat{t}\psi \right) = \left( \psi, \hat{E}\hat{t}\psi \right) - i\hbar \quad (42)$$

The things mentioned above in this section justify the next remarks

**R.15:** The Cauchy-Schwarz formula (33) is an ab origine element in respect with the usual UR (2)/(36). Moreover, (33) is always valid, independently if the conditions (34) are fulfilled or no.

**R.16:** The usual UR (2)/(36) are valid only in the circumstances strictly delimited by the conditions (34) and they are false in all other situations.

**R.17:** Due to the relations (37) and (38) in the cases described by the wave functions (6) or (17) the conditions (34) are not fulfilled. Consequently in such cases the usual UR (2)/(36) are essentially inapplicable for the pairs  $L_z - \varphi$  respectively  $N - \phi$ . However one can see that in the respective cases the Cauchy-Schwarz formula (33) remains valid as a trivial equality  $0 = 0$ .

**R.18:** In the cases of EXR described by (19) the  $L_z - \varphi$  pair satisfies the conditions (34) (mainly due to the relation (39)). Therefore in the respective cases the usual UR (2)/(36) are valid for  $L_z$  and  $\varphi$ .

**R.19:** The fulfilment of the conditions (34) by the  $L_z - \varphi$  pair for the EXR associated with (21) depends on the annulment of the right hand term in (40) (i.e. on the values of the coefficients  $c_m$ ). Adequately the correctness of the corresponding UR (2)/(36) shows the same dependence.

**R.20:** The result (41) points out the fact that the adjusted relation (10) is only a secondary piece derivable from the generally valid Cauchy-Schwarz formula (33).

**R.21:** As regards the energy-time pair the relation (42) shows that the condition (34) is never satisfied. Consequently for the respective pair the UR (2)/(36) is not applicable at all. For the same pair, described by the operators (24), the Cauchy-Schwarz formula (33) is always true. But because in QM the time  $t$  is a deterministic (i.e. non-stochastic) variable in all cases the mentioned formula degenerates into the trivial equality  $0 = 0$ .

Based on the main notifications from **R.15 - 20** now we add the following complementary remarks.

**R.22:** For the  $L_z - \varphi$  pair the relations (3) - (4) are always viable in respect with the general Cauchy-Schwarz formula (33). That is why, for a correct description of questions regarding the respective pair, it is not at all necessarily to replace the mentioned relations with some adjusted formula like [37]

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \varphi} + \alpha \quad (43)$$

or [14]

$$[\hat{L}_z, \hat{\varphi}] = -i\hbar (1 - 2\pi \delta(\varphi)) \quad (44)$$

with  $\alpha =$  an adjusting constant and  $\delta(\varphi) =$  Dirac's function of  $\varphi$ . (in (44) the notations were adapted to the stipulations  $\varphi \in [0, 2\pi)$ ,  $\hat{L}_z = -i\hbar \frac{\partial}{\partial \varphi}$  and  $\hbar \neq 1$  used in this paper).

Note that the alluded adjustments regard the cases with SCR described by the wave functions (6) when  $\varphi$  plays the role of polar coordinate. But for such a role [38] in order to be a unique (univocal) variable  $\varphi$  must be defined naturally only in the range  $[0, 2\pi)$ . (The same range is considered in practice for the normalization of the wave functions (6)). Therefore, in the cases under discussion the derivative with respect to  $\varphi$  refers to the mentioned range. Particularly for the extremities of the interval  $[0, 2\pi)$  it has to operate with backward respectively forward derivatives. So in the alluded SCR cases the relations (3) and (4) act well, with a natural correctness. The same correctness is shown by the respective relations in connection with the EXR described by the wave functions (19) or (21). In fact, from a more general perspective, the relations (3) and (4) regard the QM operators  $\hat{L}_z$  and  $\hat{\varphi}$ . Therefore they must have unique forms - i.e. expressions which do not depend on the particularities of the considered situations (e.g. systems with SCR or with EXR).

**R.23:** The troubles of UR (2) regarding  $L_z - \varphi$  and  $N - \phi$  pairs are directly connected with the conditions (34). Then it is strange that in the almost all of QM literature the respective conditions are not approached adequately. The reason seems to be related with the nowadays dominant Dirac's  $\langle bra|$  and  $|ket \rangle$  notations. In the respective notations the terms from the both sides of (34) have a unique representation namely  $\langle \psi | \hat{A}_j \hat{A}_k | \psi \rangle$ . Such a uniqueness can entail confusion (unjustified supposition) that the alluded conditions are always fulfilled. It is interesting to note that systematic investigations on the confusions/surprises generated by the Dirac's notations were started only recently [39]. Probably that further efforts on the line of such investigations will bring a new light on the conditions (34) as well as on other QM questions.

The findings from the present section give solid arguments for the following concluding remarks.

**R.24:** In respect with the conjugated observables  $L_z - \varphi$ ,  $N - \phi$  and  $E - t$  the usual UR (2)/(36) is not adequate for the role of normality standard. For such a role the Cauchy-Schwarz formula (33) is the most suitable. In some cases of interest the respective formula degenerates in the trivial equality  $0 = 0$ .

**R.25:** In reality the usual procedures of QM (illustrated by the relations (3), (4), (12), (15), (24) and (25)) work well and without anomalies in all situations regarding the above mentioned pairs of observables. Consequently in respect with the conceptual as well as practical interests of science the adjusted UR like (8), (9), (10) or (28) appear as useless inventions.

**R.26:** Conjointly the previous two remarks prove the fact that the cases of the pairs  $L_z - \varphi$ ,  $N - \phi$  and  $E - t$  infringe in an irrefutable manner the idea **BI.4** of CIUR. But such a fact must be notified as an unsurmountable shortcoming of CIUR doctrine.

## 6 A class of additional shortcomings

In the main the CIUR shortcomings discussed above in connection with the pairs  $L_z - \varphi$ ,  $N - \phi$  and  $E - t$  regard the idea **BI.4**. Besides this fact the other CIUR ideas, namely **BI.1 - 3** and **BI.5** are troubled by additional shortcomings less reported in literature. As a rule, in publications, the respective shortcomings are either ignored or mentioned on rare occasions. Moreover, even in the alluded occasions the things are presented separately but not grouped together in reunions destined for collective confrontings with CIUR. Here we attempt to put forward such a reunion.

In this attempt we focus our attention on a category of facts able to offer evidences about the mentioned additional shortcomings. The announced facts are discussed piece by piece in the following remarks.

**R.27:** First of all we note the fact that the relation (1) is improper for a reference/standard element of a supposed solid doctrine such as CIUR. This happens because the respective relations have a transitory/temporary character since they were founded on old resolution criteria (introduced by Abe and Rayleigh - see [30,40]). But the respective criteria were improved in the so-called super-resolution techniques worked out in modern experimental physics [41-48]. Then it is possible to imagine some super-resolution-thought-experiments (*srte*). So, for the corresponding *srte*-uncertainties  $\Delta_{srte}A$  and  $\Delta_{srte}B$  of two observables  $A$  and  $B$  the following relation can be promoted

$$\Delta_{srte}A \cdot \Delta_{srte}B < \hbar \quad (45)$$

Such a relation is able to replace the CIUR basic formula (1). But the alluded possibility invalidate the idea **BI.1** and incriminate CIUR in connection with one of its main points.

**R.28:** Secondly let us refer to the term *uncertainty* used by CIUR for quantities like  $\Delta_\psi A$  from (2). We think that the respective term is groundless because

of the following considerations. As it is defined in the mathematical framework of QM (see the previous section)  $\Delta_\psi A$  signifies the standard deviation of the observable  $A$  regarded as a stochastic variable. The mentioned framework deals with theoretical concepts and models about the intrinsic (inner) properties of the considered system but not with elements which refer to the measurements performed on the respective system. Consequently, for a physical system,  $\Delta_\psi A$  refers to the intrinsic characteristics (reflected in fluctuations) of the observable  $A$ . Moreover, the expressions (7), (20), (22) and (23) reveal the following realities:

- (i) for a system in a given state the quantity  $\Delta_\psi A$  has a well defined value connected with the corresponding wave function,
- (ii) the respective value of  $\Delta_\psi A$  is not related with the possible modifications of the accuracies regarding the measurement of the observable  $A$ .

The alluded realities are attested by the fact that for the same state of the measured system (i.e. for the same value of  $\Delta_\psi A$ ) the measuring uncertainties (regarding  $A$ ) can be changed through the improving or worsening of experimental devices/procedures. Note that the above mentioned realities imply and justify the observation [49] that, for two variables  $x$  and  $p$  of the same system, the usual CIUR statement "as  $\Delta x$  approaches zero,  $\Delta p$  becomes infinite and vice versa" is a doubtful speculation. Finally we can conclude that the ensemble of the things revealed in the present remark contradict the ideas **BI.2 - 4** of CIUR. But such a conclusion must be reported as a serious shortcoming of CIUR.

**R.29:** The remark **R.27** restricts the basic reference element of CIUR only to the theoretical UR (2). On the other hand, as it was pointed in Section 5 the UR (2) is nothing but a secondary and problematical piece derived from the primary Cauchy-Schwarz formula (33). Then it results that in discussions about CIUR the piece UR (2) must be referred only through its affiliation to the Cauchy-Schwarz formula. Such an affiliation discloses shortcomings of CIUR in respect with the pairs of non-commutable observables  $L_z - \varphi$ ,  $N - \phi$  and  $E - t$  (see sections 3, 4 and 5). But note that the mentioned affiliation reveals shortcomings of CIUR even in the cases of commutable observables. An example of such a case one finds with the cartesian momenta  $p_x$  and  $p_y$  for a particle in a 2D potential well. The well is delimited as follows: the potential energy  $V$  is null for  $0 < x_1 < a$  and  $0 < y_1 < b$  respectively  $V = \infty$  otherwise, where  $0 < a < b$ ,  $x_1 = (x + y)/\sqrt{2}$  and  $y_1 = (y - x)/\sqrt{2}$ . For the particle in the lowest energetic state one finds

$$\Delta_\psi p_x = \Delta_\psi p_y = \hbar \frac{\pi}{ab} \sqrt{\frac{a^2 + b^2}{2}} \quad (46)$$

$$|\langle (\delta_\psi \hat{p}_x \psi, \delta_\psi \hat{p}_y \psi) \rangle| = \left( \frac{\hbar \pi}{ab} \right)^2 \cdot \left( \frac{b^2 - a^2}{2} \right) \quad (47)$$

With these expressions it results directly that for the considered example the momenta  $p_x$  and  $p_y$  satisfy the Cauchy-Schwarz formula (33) in a non-trivial form (i.e. as an inequality with a non-null value for the right hand side term).

But such a result conflicts with the idea **BI.4** and consequently it must be reported as an element which incriminates the CIUR doctrine.

**R.30**: The UR (2)/(36) is only a particular (two-observable) version of the more general many-observable formulae (31). Then for the respective formulae CIUR has to find an interpretation concordant with its own doctrine (summarized in **BI.1 - 5**). Such an interpretation was proposed in [50] but it remained as an unconvincing thing (because of the lack of real physical justifications). Other discussions about the formulae (31) as in [13] elude any interpretation of the mentioned kind. A recent attempt [51] meant to promote an interpretation of relations like (31), for three or more observables. But the respective attempt has not a helping value for CIUR doctrine. This is because instead of consolidating the CIUR ideas **BI.1 - 5** it seems rather to support the idea that the considered relations are fluctuations formulae (in the sense discussed above and bellow in **R.28** respectively in **R.33**). We opine that to find a CIUR-concordant interpretation for the many-observable formulae (31) is a difficult (even impossible) task on natural ways (i.e. without esoteric and/or non-physical considerations). An exemplification of the respective difficulty can be appreciated by investigating the case of observables  $A_1 = L_z$ ,  $A_2 = \varphi$  and  $A_3 = H = \text{energy}$  in the situations described by the wave functions (6), (19) or (21).

**R.31**: The UR (2)/(36) fails in the case of eigenstates. The fact was mentioned in [52] but it seems to remain unremarked in the subsequent publications. In terms of the here developed investigations the alluded failure can be discussed as follows. For two non-commutable observables  $A$  and  $B$  in an eigenstate of  $A$  one obtains the set of values:  $\Delta_\psi A = 0$ ,  $0 < \Delta_\psi B < \infty$  and  $\langle [\hat{A}, \hat{B}] \rangle_\psi \neq 0$ . But, evidently, the respective values infringe the UR (2)/(36). Such situations one finds particularly with the pairs  $L_z - \varphi$  and  $N - \phi$  in cases of states described by the wave functions (6) and (17) respectively (see the Section 3). A similar example is given by the pair  $A = H = L_z^2/2I = \text{Hamiltonian (energy)}$  and  $B = \varphi$  in respect with states described by the wave functions (6).

Now one can see that the question of eigenstates does not engender any problem if the quantities  $\Delta_\psi A$  and  $\Delta_\psi B$  are regarded as fluctuations characteristics (see the remarks **R.28** and **R.33**). Then the mentioned set of values show that in the respective eigenstate  $A$  has not fluctuations (i.e.  $A$  behaves as a deterministic variable) while  $B$  is endowed with fluctuations (i.e.  $B$  appears as a stochastic variable). Note also that in the cases of specified eigenstates the UR (6)/(36) are not valid. This happens because of the fact that in such cases the conditions (34) are not satisfied. The respective fact is proved by the observation that its opposite imply the absurd result

$$a \cdot \langle B \rangle_\psi = \langle [\hat{A}, \hat{B}] \rangle_\psi + a \cdot \langle B \rangle_\psi \quad (48)$$

with  $\langle [\hat{A}, \hat{B}] \rangle_\psi \neq 0$  and  $a = \text{eigenvalue of } \hat{A}$  (i.e.  $\hat{A}\psi = a\psi$ ). But in the cases of the alluded eigenstates the Cauchy-Schwarz formula (33) remain valid. It

degenerates into the trivial equality  $0 = 0$  (because  $\delta_\psi \hat{A} \psi = 0$ ). So one finds a contradiction with **BI.4** - i.e. an additional and distinct shortcoming of CIUR.

**R.32:** Now let us note the fact UR (2)/(36) as well as the relations (31) and (33) are one-temporal formulae because all the implied quantities refer to the same instant of time. But the mentioned formulae can be generalized into multi-temporal versions, in which the corresponding quantities refer to different instants of time. So (33) is generalizable in the form

$$\Delta_{\psi_1} A \cdot \Delta_{\psi_2} B \geq \left| \left( \delta_{\psi_1} \hat{A} \psi_1, \delta_{\psi_2} \hat{B} \psi_2 \right) \right| \quad (49)$$

where  $\psi_1$  and  $\psi_2$  represent the wave function for two different instants of time  $t_1$  and  $t_2$ . If in (49) one takes  $|t_2 - t_1| \rightarrow \infty$  in the CIUR vision the quantities  $\Delta_{\psi_1} A$  and  $\Delta_{\psi_2} B$  have to refer to  $A$  and  $B$  regarded as independent solitary observables. But in such a regard if  $\left( \delta_{\psi_1} \hat{A} \psi_1, \delta_{\psi_2} \hat{B} \psi_2 \right) \neq 0$  the relation (49) refute the idea **BI.2** and so it reveals another additional shortcoming of CIUR. Note here our opinion that the various attempts [53], [54] of extrapolating the CIUR vision onto the relations of type (49) are nothing but artifacts without any real (physical) justification. We think that the relation (49) does not engender any problem if it is regarded as fluctuations formula (in the sense mentioned in **R.28** and **R.33**). In such a regard the cases when  $\left( \delta_{\psi_1} \hat{A} \psi_1, \delta_{\psi_2} \hat{B} \psi_2 \right) \neq 0$  refer to the situations in which, for the instants  $t_1$  and  $t_2$ , the corresponding fluctuations of  $A$  and  $B$  are correlated (i.e. statistically dependent).

**R.33:** Now let us call attention on a quantum-classical similarity which directly contradicts the idea **BI.5** of CIUR. The respective similarity directly regards the UR (2)/(36) as descendant from the relations (33) and (31). Indeed the mentioned relations are completely analogous with a set of classical formulas from phenomenological theory of fluctuations. The alluded formulae can be written [55], [56] as follows

$$\det \left[ \langle \delta_w A_j \delta_w A_k \rangle_w \right] \geq 0 \quad (50)$$

$$\Delta_w A \cdot \Delta_w B \geq \left| \langle \delta_w A \delta_w B \rangle_w \right| \quad (51)$$

In these formulae  $A_j$ ,  $A$  and  $B$  signify the classical global observables which characterize a thermodynamic system in its wholeness. In the same formulae  $w$  denotes the phenomenological probability distribution,  $\langle \dots \rangle_w$  represents the mean (expected value) evaluated by means of  $w$  while  $\Delta_w A$ ,  $\Delta_w B$  and  $\langle \delta_w A \delta_w B \rangle_w$  stand for characteristics (standard deviations respectively correlation) regarding the fluctuations of the mentioned observables. We remind the appreciation that in classical physics the alluded characteristics and, consequently, the relations (50) - (51) describe the intrinsic (own) properties of thermodynamic systems but not the aspects of measurements performed on the respective systems. Such an appreciation is legitimated for example by the research regarding the fluctuation spectroscopy [57] where the properties



of macroscopic (thermodynamic) systems are evaluated through the (spectral components of) characteristics like  $\Delta_w A$  and  $\langle \delta_w A \delta_w B \rangle_w$ .

The above discussions disclose the groundlessness of idea [58]- [60] that the relations like (51) have to be regarded as a sign of a macroscopic/classical complementarity (similar with the quantum complementarity supposed by CIUR idea **BI.4**). According to the respective idea the quantities  $\Delta_w A$  and  $\Delta_w B$  appear as macroscopic uncertainties. Note that the mentioned idea was criticized partially in [61, 62] but without any explicit specification that the quantities  $\Delta_w A$  and  $\Delta_w B$  are characteristics of fluctuations.

The previously notified quantum-classical similarity together with the reminded significance of the quantities implied in (50) and (51) suggests and consolidates the following regard (argued also in **R.28**). The quantities  $\Delta_\psi A$  and  $\Delta_\psi B$  from UR (2)/(36) must be regarded as describing intrinsic properties (fluctuations) of quantum observables  $A$  and  $B$  but not as uncertainties of such observables.

Now, in conclusion, one can say that the existence of classical relations (50) and (51) contravenes to both ideas **BI.5** and **BI.1** of CIUR.

**R.34:** In classical physics the fluctuations of  $A$  and  $B$  implied in (51) are described not only by the second order parameters like  $\Delta_w A$ ,  $\Delta_w B$  or  $\langle \delta_w A \delta_w B \rangle_w$ . For a better evaluation the respective fluctuations are characterized additionally [63] by higher order moments like  $\langle (\delta_w A)^r (\delta_w B)^s \rangle_w$  with  $r + s \geq 3$ . This fact suggests the observation that, in the context considered by CIUR, we also have to use the quantum higher order moments like  $\langle (\delta_\psi A)^r (\delta_\psi B)^s \rangle_\psi$  with  $r + s \geq 3$ . Then for the respective quantum moments CIUR is obliged to offer an interpretation compatible with its own doctrine. But it seems to be less probable that such an interpretation can be promoted through credible (and natural) arguments.

**R.35:** The thermodynamic systems were also implied in other debates about CIUR, in connection with the question of the so called "macroscopic operators" (see [64], [65] and references). The question appeared as follows. By analogy with UR (2) and viewing the respective systems in terms of quantum statistical physics, the CIUR partisans promoted the formula

$$\Delta_\rho A \cdot \Delta_\rho B \geq \frac{1}{2} \left| \left\langle \left[ \hat{A}, \hat{B} \right] \right\rangle_\rho \right| \quad (52)$$

This formula implies the notations:  $A$  and  $B$  denote two global observables (for the system in its wholeness) described by the corresponding operators  $\hat{A}$  and  $\hat{B}$ ,  $\hat{\rho}$  signifies the statistical operator (density matrix) associated with the global state of the system, respectively  $\Delta_\rho A = \left\{ Tr \left[ \left( \hat{A} - \langle A \rangle_\rho \right)^2 \hat{\rho} \right] \right\}^{\frac{1}{2}}$  where  $\langle A \rangle_\rho = Tr \left( \hat{A} \hat{\rho} \right) =$  the mean value of  $A$ . Relation (52) entailed discussions because of the conflict between the following two findings:

(i) On one hand (52) is introduced by analogy with UR (2)/(36) on which CIUR is founded. Then, by extrapolating CIUR, the quantities  $\Delta_\rho A$  and  $\Delta_\rho B$

from (52) should be interpreted as (global) uncertainties subjected to stipulations as the ones indicated in **BI.3** and **BI.4**.

(ii) On the other hand, in the spirit of **BI.5**, CIUR agrees with the idea that the observables characterizing the thermodynamic systems are possible to be measured without any uncertainty (i.e. with unbounded accuracy). For an observable the mentioned possibility should be independent of the fact that it is measured solitarily or simultaneously with other observables. Thus, for two thermodynamic observables, it is senselessly to accept stipulations such are the ones prescribed by **BI.3** and **BI.4**.

In order to elude the mentioned conflict a strange purpose was promoted, namely: to abrogate the formula (52) and to replace it with an adjusted macroscopic relation concordant with CIUR vision. For such a purpose the global operators  $\hat{A}$  and  $\hat{B}$  from (52) were substituted [64], [65] by the so-called "macroscopic operators"  $\hat{\mathcal{A}}$  and  $\hat{\mathcal{B}}$ . The respective "macroscopic operators" are considered to be representable as quasi-diagonal matrices (i.e. as matrices with non-null elements only in a "microscopic neighbourhood" of principal diagonals). Then one supposes that  $[\hat{\mathcal{A}}, \hat{\mathcal{B}}] = 0$  for any pairs of observables A and B and, consequently instead of (52) one obtains

$$\Delta_\rho \hat{\mathcal{A}} \cdot \Delta_\rho \hat{\mathcal{B}} \geq 0 \quad (53)$$

In this formula CIUR partisans see the fact that the uncertainties  $\Delta_\rho A$  and  $\Delta_\rho B$  can be unboundedly small at the same instant of time. Such a fact is in concordance with CIUR vision about macroscopic observables. Today it seems to be accepted the belief that the adjusted relation (53) solves all the troubles of CIUR caused by the formula (52).

A first disapproval of the mentioned belief results from the following observations:

(i) Relation (52) cannot be abrogated if the entire mathematical apparatus of quantum statistical physics is not abrogated too. More exactly, the substitution of operators from the global version  $\hat{A}_j$  into a "macroscopic" variant  $\hat{\mathcal{A}}_j$  is a senseless invention as long as in practical procedures of quantum statistical physics [66], [67] as lucrative operators one uses  $\hat{A}_j$  but not  $\hat{\mathcal{A}}_j$ .

(ii) The substitution  $\hat{A}_j \rightarrow \hat{\mathcal{A}}_j$  does not metamorphose automatically (52) into (53), because if two operators are quasi-diagonal, in sense required by the partisans of CIUR, it is not surely that they commute. As an example we quote the Cartesian components of the global magnetization  $\vec{M}$  of a paramagnetic system formed of  $N$  independent  $\frac{1}{2}$  - spins. The alluded components are described by the global operators

$$\hat{M}_\alpha = \frac{\gamma \hbar}{2} \hat{\sigma}_\alpha^{(1)} \oplus \frac{\gamma \hbar}{2} \hat{\sigma}_\alpha^{(2)} \oplus \dots \oplus \frac{\gamma \hbar}{2} \hat{\sigma}_\alpha^{(N)} \quad (54)$$

where  $\alpha = x, y, z$ ;  $\gamma =$  magneto-mechanical factor and  $\hat{\sigma}_\alpha^{(i)}$  = Pauli matrices associated to the  $i$ -th spin (particle). Note that the operators (54) are quasi-diagonal in the mentioned sense but, for all that, they do not commute because  $[\hat{M}_\alpha, \hat{M}_\beta] = i\hbar\gamma \epsilon_{\alpha\beta\mu} \hat{M}_\mu$  ( $\epsilon_{\alpha\beta\mu}$  denote the Levi-Civita tensor).

A second disproof of the belief induced by the substitution  $\hat{A}_j \rightarrow \hat{\mathcal{A}}_j$  is evidenced if the relation (52) is regarded in an ab origine (and natural) approach similar with the one presented in Section 5. In such regard it is easy to see that in fact the formula (52) is only a restrictive descendant from the generally valid relation

$$\Delta_\rho A \cdot \Delta_\rho B \geq \left| \left\langle \delta_\rho \hat{A} \delta_\rho \hat{B} \right\rangle_\rho \right| \quad (55)$$

where  $\delta_\rho \hat{A} = \hat{A} - \langle A \rangle_\rho$ .

This last relation justifies the following affirmations:

(i) Even in the situations when  $[\hat{A}, \hat{B}] = 0$  the product  $\Delta_\rho A \cdot \Delta_\rho B$  can be lower bounded by a non-null quantity. This happens because it is possible to find cases in which the term from the right hand side of (55) has a non-null value.

(ii) Relations (55) remain valid, without any problem, even after the substitution  $\hat{A}_j \rightarrow \hat{\mathcal{A}}_j$ . Then according to the previous affirmation the respective substitution does not guarantee the relation (53) and the corresponding speculations.

The above presented facts warrant the conclusion that the relation (52) reveal a real shortcoming of CIUR. The respective shortcoming cannot be avoided by restoring to the so-called "macroscopic operators". But note that the same relation does not cause any problem if it is considered together with (55) as formulae which refer to the fluctuations of global observables regarding thermodynamic systems.

**R.36:** The quantum-classical similarity revealed in **R.33** also entails a proof against the CIUR assertion from **BI.5** that Planck constant  $\hbar$  has no analog in non-quantum physics. Such a proof results from the following facts. The alluded similarity regards the groups of relations (31), (33), (36)/(2) and (50), (51). The respective relations imply the standard deviations  $\Delta_\psi A_j$  or  $\Delta_w A_j$  associated with the fluctuations of the corresponding observables. But mathematically the standard deviation indicate the stochasticity (randomness) of a variable, in the sense that it has a positive or null value as the respective variable is a stochastic or, alternatively, deterministic (non-stochastic) quantity. Therefore the deviations  $\Delta_\psi A_j$  and  $\Delta_w A_j$  can be regarded as similar indicators of stochasticity for the quantum respectively classical observables.

For diverse cases (of observables, systems and states) the classical deviations have various expressions in which, apparently, no common element seems to be implied. Nevertheless such an element can be found out [68] as being materialized by the Boltzmann constant  $k_B$ . So, in the framework of phenomenological theory of fluctuations (in Gaussian approximation) one obtains [68]

$$(\Delta_w A_j)^2 = k_B \sum_\alpha \sum_\beta \frac{\partial \bar{A}_j}{\partial X_\alpha} \frac{\partial \bar{A}_j}{\partial X_\beta} \left( \frac{\partial^2 \bar{S}}{\partial X_\alpha \partial X_\beta} \right)^{-1} \quad (56)$$

In this relation  $\bar{A}_j = \langle A_j \rangle_w$ ,  $S = S(X_\alpha)$  denotes the entropy written as a

function of independent variables  $X_\alpha$ , ( $\alpha = 1, 2, \dots, r$ ) and  $(a_{\alpha\beta})^{-1}$  represent the elements of the inverse of matrix  $[a_{\alpha\beta}]$ . Then from (56) it results that the expressions for  $(\Delta_w A_j)^2$  consist of products of  $k_B$  with factors which are independent of  $k_B$ . The respective independence is evidenced by the fact that the alluded factors ought to coincide with deterministic (non-stochastic) quantities from usual thermodynamics. Or it is known that such quantities do not imply  $k_B$  at all. Concrete exemplifications of the relations (56) with the above noted properties are quoted in [68].

Then, as a first aspect, from (56) it results that the fluctuations characteristics (dispersions)  $(\Delta_w A_j)^2$  are directly proportional to  $k_B$  and, consequently, they are non-null respectively null quantities as  $k_B \neq 0$  or  $k_B \rightarrow 0$ . (Note that because  $k_B$  is a constant the limit  $k_B \rightarrow 0$  means that the quantities directly proportional with  $k_B$  are negligible comparatively with other quantities of same dimensionality but independent of  $k_B$ ). On the other hand, the second aspect (mentioned also above) is the fact that  $\Delta_w A_j$  are particular indicators of classical stochasticity. Conjointly the two mentioned aspects show that  $k_B$  has the qualities of an authentic generic indicator of thermal stochasticity which is specific for classical macroscopic systems. (Add here the observation that the same quality of  $k_B$  can be revealed also [68] if the thermal stochasticity is studied in the framework of classical statistical mechanics).

Now let us discuss about the quantum stochasticity whose indicators are the standard deviations  $\Delta_\psi A_j$ . Based on the relations (20) and (46) one can say that in many cases the expressions for  $(\Delta_\psi A_j)^2$  consist in products of Planck constant  $\hbar$  with factors which are independent of  $\hbar$ . Then, by analogy with the above discussed classical situations,  $\hbar$  places itself in the posture of generic indicator for quantum stochasticity.

In the alluded posture the Planck constant  $\hbar$  has an authentic classical analog represented by the Boltzmann constant  $k_B$ . But such an analogy contradicts strongly the idea **BI.5**.

In connection with the roles of  $k_B$  and  $\hbar$  as generic indicators of stochasticity it is of interest to add here the following aspect. In their above presented roles  $k_B$  and  $\hbar$  regard the one-fold stochasticity, of classical and quantum nature respectively, evaluated through the deviations  $\Delta_w A_j$  and  $\Delta_\psi A_j$ . But in physics is also known a two-fold stochasticity, of a combined thermal and quantum nature. Such a stochasticity appears in cases of quantum statistical systems and it is evaluated through the standard deviations  $\Delta_\rho A_j$  implied in relations (52) and (55). The expressions of the mentioned deviations can be obtained by means of the fluctuation-dissipation theorem [66]. Therefore

$$(\Delta_\rho A_j)^2 = \frac{\hbar}{2\pi} \int_{-\infty}^{\infty} \coth\left(\frac{\hbar\omega}{2k_B T}\right) \chi''_{jj}(\omega) d\omega \quad (57)$$

Here  $\chi''_{jj}(\omega)$  denote the imaginary parts of the susceptibilities associated with the observables  $A_j$ . Note that  $\chi''_{jj}(\omega)$  are the deterministic quantities which appear also in non-stochastic framework of macroscopic physics [69]. That is

why  $\chi''_{jj}(\omega)$  are independent of both  $k_B$  and  $\hbar$ . Then from (57) it results that  $k_B$  and  $\hbar$  considered together appear as a couple of generic indicators for the twofold stochasticity of thermal and quantum nature. The respective stochasticity is negligible when  $k_B \rightarrow 0$  and  $\hbar \rightarrow 0$  and significant when  $k_B \neq 0$  and  $\hbar \neq 0$  respectively.

The above discussions about the quantum stochasticity and the limit  $\hbar \rightarrow 0$  must be supplemented with the following specifications. The respective stochasticity regards the cases of observables of orbital and spin types respectively. In the orbital cases the limit  $\hbar \rightarrow 0$  is usually associated with the quantum  $\rightarrow$  classical limit. The respective limit implies an unbounded growth of the values of some quantum numbers so as to ensure a correct limit for the orbital movements. Then one finds [70,71] that the orbital-type stochasticity is in one of the following two situations:

- (i) in the mentioned limit it converts oneself in a classical-type stochasticity of the corresponding observables (e.g. in the cases of  $\varphi$  and  $L_z$  of a torsional pendulum or of  $x$  and  $p$  of a rectilinear oscillator), or
- (ii) in the same limit it disappears, the corresponding observables becoming deterministic classical variables (e.g. in the case of the distance  $r$  of the electron in respect with the nucleus in a hydrogen atom).

The quantum stochasticity of spin-type regards the spin observables. In the limit  $\hbar \rightarrow 0$  such observables disappear completely (i.e. they lose both their mean values and the affined fluctuations).

We end here this section where the remarks **R.27 - 36** point out a class of less discussed shortcomings of CIUR. The respective class supplements the set of the more known of CIUR defects discussed in sections 3, 4, and 5. In its wholeness the resulting set contradicts indubitably the ensemble of all CIUR basic ideas **BI.1 - 5**.

## 7 A first reconsideration of the things: Abandonment of CIUR

In sections 3 - 6 we have presented a set of shortcomings whose ensemble contradicts in an indubitable manner all the basic ideas **BI.1 - 5** of CIUR. Of course, the respective presentation ought to be supplemented with specifications regarding both the gravity of the things and the possible reconsideration of the discussed questions. In this sense we note the following remarks.

**R.37:** The mentioned contradictions are irrefutable for CIUR doctrine in the sense that they can not be surmounted by inner arguments (deductible from **BI.1 - 5**) of the respective doctrine. Consequently the vexed question about CIUR must be approached constructively by taking into account the alluded shortcomings and by looking for adequate reappraisal of the facts.

**R.38:** In the mentioned circumstances CIUR proves oneself to be an incorrect amalgam (of suppositions) deprived of necessary qualities of a valid scientific construction. That is why CIUR must be abandoned as a wrong doctrine which,

in fact, has no real value.

**R.39:** The alluded abandonment has to be completed by a natural re-interpretation of the basic CIUR's relations (1) and (2). We opine that the respective re-interpretation is argued mainly in the remarks **R.27**, **R.28** and **R.33** from Section 6. So the relations (1) appear as fictions without any physical significance. On the other hand the relations (2) are simple fluctuations formulae, from the same family with the microscopic and macroscopic relations from the groups (31), (33) and (36) respectively (50), (51) and (55). Consequently the relations (1) and (2) have no special or extraordinary status/significance in physics.

**R.40:** The reappraisals noted in **R.38** and **R.39** do not disturb in any way the framework (conceptions and procedures) of usual QM as it is applied concretely in the investigations of quantum systems.

**R.41:** The above alluded reappraisals disconnect the relations (1) and (2) from the description of quantum measurements. So the respective description become a distinct scientific question. It will be discussed in the next sections.

## 8 An inspection of the conventional views about quantum measurements

The question regarding the description of the quantum measurements (QMS) is one of the most debated subject associated with the CIUR history. It generated a large diversity of viewpoints relatively to its importance and/or approach (see [1, 25–29] and references). The respective diversity inserts even some extreme opinions such are:

- (i) the description of QMS is *"probably the most important part of the theory ("QM")*" [1].
- (ii) *"the word ("measurement") has had such a damaging effect on the discussions that ... it should be banned altogether in quantum mechanics"* [72].

As a notable aspect today one finds that the many of the existing approaches of QMS (including some of the most recent ones) are of conventional essence. This happens because they are grounded on some conventional premises (**CP**) which presume and even try to extend the CIUR doctrine (see [1, 6, 25–29, 73–75] and references). That is why, a reconsideration of CIUR like the one presented above in the previous sections, requires a corresponding inspection of the mentioned **CP** regarding the QMS. We start such an inspection by pointing out the fact that, in essence, the alluded **CP** can be resumed as follows:

**CP.1 (basic):** *The descriptions of QMS must be developed as confirmations and extensions of CIUR doctrine.*

**CP.2 (supporting CIUR):** *The peculiarities of QMS reported in BI.2 -4 are connected with the corresponding features of the measuring perturbations (which trouble the investigated systems during the measurements). So in the cases of observables refered in BI.2 - 3 respectively in BI.4 the alluded perturbations are supposed to have an avoidable respectively an unavoidable character.*

**CP.3** (*supporting CIUR*): In the case of QMS the mentioned perturbations cause specific jumps in states of the measured systems. The respective jumps have to be included obligatory in the descriptions of QMS.

**CP.4** (*supporting CIUR*): With regard to the observables of quantum and classical type respectively the measuring inconveniences (perturbations and uncertainties) show an essential difference. Namely they are unavoidable respectively avoidable characteristics of measurements. The mentioned difference must be taken into account as a main point in the descriptions of the measurements regarding the two types of observables.

**CP.5** (*extending CIUR*): For a quantum observable  $A$  of a system in the state  $|\psi\rangle$  a QMS is assumed to give as result a single value say  $a_n$  which is one of the eigenvalues of the associated operator  $\hat{A}$ . Therefore the description of the respective QMS must include as essential piece a sudden reduction (collapse) of the wave function i.e a relation of the form:

$$|\psi(t)\rangle \text{ before measurement} \rightarrow |a_n\rangle \text{ after measurement} \quad (58)$$

where  $t$  denotes the instant of QMS and  $|a_n\rangle$  represents the eigenfunction of  $\hat{A}$  corresponding to the eigenvalue  $a_n$ . **CP.6** (*extending CIUR*): The description of QMS ought to be incorporated as an inseparable part in the framework of QM. Adequately QM must be considered as a unitary theory both of intrinsic properties of quantum systems and of measurements regarding the respective properties.

Now the announced inspection of the conventional conceptions about QMS can be focused in comments on the above premises **CP.1 - 6**. In the spirit of the previous discussions regarding CIUR for the alluded comments of prime importance is to note that the premises **CP.1 - 6** are troubled by many shortcomings. The respective troubles are pointed out piece by piece through the following remarks.

**R.42**: As we concluded in Section 7 in fact CIUR is nothing but a wrong doctrine which must be abandoned. Consequently CIUR has to be omitted from the lucrative scientific discussions. Moreover it is illegitimately to approach a scientific question (as is the description of QMS) by using and extending such a doctrine. That is why the premise **CP.1** is totally groundless.

**R.43**: The premise **CP.2** is inspired and argued by the ideas of CIUR about the relations (1) and (2). But, according to the discussions from the previous sections, the respective ideas are completely unfounded. Therefore the alluded **CP.2** is deprived of any necessary and well-grounded justification.

**R.44**: In the main **CP.3** is inferred from the belief that the mentioned jumps have an essential importance for QMS. But the respective belief appears as entirely unjustified if one takes into account the following natural and indubitable observation [76]: "it seems essential to the notion of measurement that it answers a question about the given situation existing before the measurement. Whether the measurement leaves the measured system unchanged or brings about a new and different state of that system is a second and independent question". So we have to report an inevitable deficiency of **CP.3**.

**R.45:** The essence of the difference mentioned in **CP.4** is questionable at least because of the following two reasons:

(i) In the classical case the mentioned avoidance of the measuring inconveniences have not a significance of principle but only a relative and limited value (depending on the performances of measuring devices and procedures). Such a fact seems to be well known by experimenters.

(ii) In the quantum case until now the alluded unavoidableness cannot be justified by valid arguments of experimental nature (see the above remark **R.27** and the comments regarding the relation (45)).

**R.46:** Through the assumption implied in **CP.5** a QMS is presumed to consist of a single experimental trial. But as it is well known the quantum observables are stochastic variables endowed with specific spectra of values. On the other hand, from a mathematical perspective [36], for a stochastic variable a single experimental trial (outcome) has no significance. A true measurement (experimental evaluation) of such a variable requires a statistical sampling composed by a (large) number of single trials. The results of the mentioned sampling facilitate the estimation of the probabilistic characteristics (e.g. mean value and standard deviation) of the respective variable. The mentioned features of measurements regarding stochastic observables are taken into account in the classical (non-quantum) context in connection with the phenomenological theory of fluctuations [77, 78]. In the respective context a stochastic observable  $A$  is characterized [55, 56, 79] by a probability distribution  $w(a)$ . But for a true measurement of  $A$  a single experimental trial, which gives a unique value - say  $a_0$ , has no significance. Consequently, there it is no interest for a reduction (collapse) of probability distribution like

$$w(a) \quad \text{before measurement} \rightarrow \delta(a - a_0) \quad \text{after measurement} \quad (59)$$

with  $\delta =$  Dirac function.

Then by a credible analogy one can say that in respect with QMS the reduction/collapse (58) has not any real scientific meaning.

**R.47:** The reduction/collapse (58) also appears as meaningless if the question of QMS is regarded from the perspective of Albertson's observation [76] quoted above in **R.44**.

**R.48:** The premise **CP.6** proves to be an unjustified idea if the usual conventions of physics are considered. According to the respective conventions, in all the basic chapters of physics, each observable of a system is regarded as a concept "*per se*" (in its essence) which is denuded of measuring aspects. Or QM is nothing but such a basic chapter, like classical mechanics, thermodynamics, electrodynamics or statistical physics. On the other hand in physics the measurements appear as main purposes of experiments. But note that the study of the experiments has its own problems [80] and is done in frameworks which are additional and distinct in respect with the basic chapters of physics. The above note is consolidated by the observation that [81]: "*the procedures of measurement (comparison with standards) has a part which cannot be described inside the branch of physics where it is used*".



Then, in contrast with the premise **CP.6**, it is natural to accept the idea that QM and the description of QMS have to remain distinct scientific branches. However the two branches have to use some common concepts and symbols. This happens because, in fact, both of them also imply elements regarding the same quantum systems.

In the end of this section in can be seen that remarks **R.42 - 48** point out serious shortcomings of all premises **CP.1 - 6** regarding QMS. Consequently the conventional approaches of QMS prove themselves to be unsuccessful endeavours. Then, with regard to QMS, it can be of some nontrivial interest to search for possible new approaches, dissociated from the premises **CP.1 - 6**. Such an approach is presented in the next sections.

## 9 A second reconsideration of things: New views about QMS

It is known that the above presented premises **CP.1 - 6** persist in nowadays publications regarding the problem of QMS description. Then, by taking into account the remarks **R.42 - 48** from the previous section, it results that the mentioned problem is still an open question (at least partially). The respective question requires reconsidered approaches founded on new premises (**NP**) and disconnected of conventional premises **CP.1 - 6** and CIUR doctrine. We attempt to present such **NP** in this section.

Firstly we note that a natural theory of measurements must contain elements (concepts and reasonings) which are in adequate correspondence with the main characteristics of the real measuring experiments. Then, for our attempt, it is of major interest to account the alluded characteristics of QMS regarded in the general context of scientific practice (with its proved and accepted views). We start the mentioned account with the observation that in classical physics the belief [82] *"in the objective existence of material systems ... which possess properties independently of measurements"* is accepted as an axiom. The respective axiom implies the idea that a measurement aims to give information about the pre-existent state of the investigated system. We opine that the mentioned axiom and idea must also be adopted in connection with the quantum systems. Our opinion is encouraged by the following two remarks expressed with regard to QMS:

(i) *"when it is said that something is "measured" it is difficult not to think of the results as referring to some preexisting property of the object in question"* [72].

(ii) *"the function of measurement in quantum mechanics is to determine the average value and the dispersion of some physical quantity in a given system as they are prior the measurement"* [76].

For discussions on the QMS it is also important to underline the fact that quantum observables are stochastic variables. Then, as it was pointed out in **R.46**, in the last analysis a true measurement of such an observable requires

a statistical sampling (composed from a large number of single experimental trials). A similar requirement is found in the case of measurements regarding the measurement of classical (non-quantum) observables with stochastic characteristics. That is why we think that in the description of QMS there can be much interest in some ideas referring to the measurements of the alluded classical observables (like the ones discussed in [77, 78] ).

The above considerations plead for the idea that for the description of QMS a naturally reconsidered approach can be founded on the following new premises (**NP**):

**NP.1:** *The real purpose of a QMS is to give information about the pre-existent state of the investigated system. Therefore for a natural description of a QMS it is needlessly any reference to a collapse (reduction) of the respective state into a post-measurement one.*

**NP.2:** *The description of QMS has to assimilate some ideas regarding the measurements of classical stochastic observables. This is because, in practice, both types of measurements consist in similar statistical samplings.*

**NP.3:** *Since QMS refer to the systems studied in QM their descriptions ought to use some QM concepts (e.g. wave functions and operators).*

**NP.4:** *The procedures of QMS include parts which do not belong to QM. Therefore the description of QMS have to be regarded not as a part of QM-theory but as a distinct scientific branch.*

By taking into account the above premises **NP.1 - 4** a new approach of QMS description can be developed, which will be presented below in the next sections.

## 10 A suggesting classical model

An approach of QMS, reconsidered in the above mentioned senses, can be started with some discussions about a suggesting model which regards the measurements of classical stochastic observables. For such a model we refer to the case of an observable A appearing in the phenomenological theory of fluctuations [65], [79] (mentioned also in the remark **R.46** ). The values  $a$  of such an observable range in a continuous and infinite spectrum (i.e.  $a \in (-\infty, \infty)$ ). The respective values are associated with the probability distribution  $w(a)$ . In respect with the values of A a measurement can be regarded [76], [78] as an input  $\rightarrow$  output (*in*  $\rightarrow$  *out*) process which imply the transformation

$$w_{in}(a) \rightarrow w_{out}(a) \tag{60}$$

In the alluded regard  $w_{in}(a)$  refers to the intrinsic values of A, associated with the inner properties of the measured system, while  $w_{out}(a)$  is related with the output values of A, displayed on the recorder of the measuring device. Note that, without any loss of generality, the *in* and *out* spectra of A (associated to  $w_{in}(a)$  and  $w_{out}(a)$ ) can be considered as coincident. By means of  $w_{\eta}(a)$  ( $\eta = in, out$ ) the corresponding (numerical) characteristics of A regarded as stochastic variable can be introduced. In the spirit of usual practice of physics

we refer here only to the two lowest order such characteristics. They are the  $\eta$  - mean (expected) values  $\langle A \rangle_\eta$  and  $\eta$  - standard deviations  $\Delta_\eta A$  defined as follows

$$\langle A \rangle_\eta = \int_{-\infty}^{\infty} a w_\eta(a) da \quad (\Delta_\eta A)^2 = \left\langle \left( A - \langle A \rangle_\eta \right)^2 \right\rangle_\eta \quad (61)$$

The values of  $A$  displayed by the measuring device are evaluated (by sampling and processing) according to the rules of mathematical statistics [36, 83]. As regards the observable  $A$  the mentioned evaluation attends to provide optimal estimators for the *out*-characteristics like  $\langle A \rangle_{out}$  and  $\Delta_{out} A$  (or even for the *out*-distribution  $w_{out}(a)$ ). It is a fact that some publications seem to admit (tacitly) the idea that the alluded evaluation takes into account the *in*-characteristics of  $A$  (like  $\langle A \rangle_{in}$ ,  $\Delta_{in} A$  or even  $w_{in}(a)$ ). But one can see that such an idea is justified only in the case of ideal measurements when  $w_{out}(a) = w_{in}(a)$ . However in most of the real situations the measurements are non-ideal and  $w_{out}(a) \neq w_{in}(a)$ . Consequently we think that, for the considered measurements of  $A$ , it is necessary to search a theoretical description able to give a concrete and credible expression for the transformation (60). In the spirit of our works [77], [78] the respective search can be materialized by taking into account the following aspects:

- (i) For a characterization of the measuring device we use the transfer probability  $G(a, a')$  with the significance : (i.1)  $G(a, a')da$  denote the (infinitesimal) probability that by measurement the *in*-value  $a'$  of  $A$  to be recorded in the *out*-interval  $(a, a + da)$ , (i.2)  $G(a, a')da'$  represents the probability that the *out*-value  $a$  to result from the *in*-values which belong to the interval  $(a', a' + da')$ .
- (ii) The stochastic characteristics of the measuring device and of the measured system respectively are completely independent.

According to the rules of composition for probabilities the transformation (60) can be written as

$$w_{out}(a) = \int_{-\infty}^{\infty} G(a, a') w_{in}(a') da' \quad (62)$$

Due to the significances mentioned above in (i.1) and (i.2) the kernel  $G(a, a')$  satisfies the relation

$$\int_{-\infty}^{\infty} G(a, a') da = \int_{-\infty}^{\infty} G(a, a') da' = 1 \quad (63)$$

One observes that in (62)  $G(a, a')$  describe the characteristics of the measuring devices. Particularly  $G(a, a') = \delta(a - a')$  and  $G(a, a') \neq \delta(a - a')$  (where  $\delta$  denotes the Dirac function) describe an ideal respectively a non-ideal device. Differently  $w_{in}(a')$  refers to the properties of the measured system. So  $w_{out}(a)$  incorporate information regarding both the mentioned device and system.

Now, from the general perspective of the present paper, it is of interest to note some observations about the measuring uncertainties (errors). Firstly it is important to remark that for the discussed observable  $A$ , the standard deviations  $\Delta_{in}A$  and  $\Delta_{out}A$  are not estimators of the mentioned uncertainties. Of course that the above remark contradicts some loyalties induced by CIUR doctrine. Here it must be pointed out that:

- (i) On one hand  $\Delta_{in}A$  together with  $\langle A \rangle_{in}$  describe only the intrinsic properties of the measured system.
- (ii) On the other hand  $\Delta_{out}A$  and  $\langle A \rangle_{out}$  incorporate composite information about the respective system and the measuring device.

Then, in terms of the above considerations, the measuring uncertainties of  $A$  are described by the following error indicators (characteristics)

$$\varepsilon \{ \langle A \rangle \} = | \langle A \rangle_{out} - \langle A \rangle_{in} | , \quad \varepsilon \{ \Delta A \} = | \Delta_{out}A - \Delta_{in}A | \quad (64)$$

Note that because  $A$  is a stochastic variable for an acceptable evaluation of its measuring uncertainties it is completely insufficient the single indicator  $\varepsilon \{ \langle A \rangle \}$ . Such an evaluation requires at least the couple  $\varepsilon \{ \langle A \rangle \}$  and  $\varepsilon \{ \Delta A \}$  or even the differences of the higher order moments like

$$\varepsilon \{ \langle (\delta A)^n \rangle \} = | \langle (\delta_{out}A)^n \rangle_{out} - \langle (\delta_{in}A)^n \rangle_{in} | \quad (65)$$

where  $\delta_\eta A = A - \langle A \rangle_\eta ; \eta = in, out ; n \geq 3$ .

Add here the observation that a comprehensive characterization of the measuring uncertainties regarding  $A$  can be done [77] in terms of informational (Shannon) entropies  $S(w_\eta)$  defined as

$$S(w_\eta) = - \int_{-\infty}^{\infty} w_\eta(a) \ln w_\eta(a) da, \quad (\eta = in, out) \quad (66)$$

By using the relations (62) and (63) together with the evident formula  $x - 1 \geq \ln(x)$  (with  $x \in (0, \infty)$ ) it is easy to prove [77] the following result

$$\varepsilon \{ S(w) \} = S(w_{out}) - S(w_{in}) \geq 0 \quad (67)$$

This result shows that during the measurement the error  $\varepsilon \{ S(w) \}$  of the informational entropy (associated with the observable  $A$ ) is a real and non-negative quantity. In the last part of (67) the signs = and > correspond to an ideal and a non-ideal measurement respectively.

## 11 The suggested quantum model

Now let us develop a reconsidered model for description of QMS. The announced development conforms oneself to the premises **NP.1 - 4** and assimilates some ideas suggested by the classical model discussed in the previous section. We restrict our model only to the measurements of quantum observables of orbital

nature (i.e. coordinates, momenta, angles, angular momenta and energy). The respective observables are described by the operators  $\hat{A}_j$  ( $j = 1, 2, \dots, n$ ) regarded as generalized stochastic variables. As a measured system we consider a spinless microparticle whose state is described by the wave function  $\psi = \psi(\vec{r}, t)$ , taken in the coordinate representation ( $\vec{r}$  and  $t$  stand for microparticle's position and time respectively). Account here the fact that, because we consider only a non-relativistic context, the explicit mention of time  $t$  in the expression of  $\psi$  is unimportant.

Now note the observation that the wave function  $\psi(\vec{r})$  incorporate information (of probabilistic nature) about the measured system. That is why a QMS can be regarded as a process of information transmission: from the respective system to the recorder of the measuring device. Then, on the one hand, the input (*in*) information described by  $\psi_{in}(\vec{r})$  refers to the intrinsic (own) properties of the measured system (regarded as information source). The expression of  $\psi_{in}(\vec{r})$  is deducible within the framework of usual QM (e.g. by solving the adequate Schrodinger equation). On the other hand the output (*out*) information, described by the wave function  $\psi_{out}(\vec{r})$ , refers to the data obtained on the device recorder (regarded as information receiver). So the measuring device plays the role of the transmission channel for the alluded information. Accordingly the measurement appears as a processing information operation. By regarding the things as above the description of the QMS must be associated with the transformation

$$\psi_{in}(\vec{r}) \rightarrow \psi_{out}(\vec{r}) \quad (68)$$

As in the classical model (see the previous section), without any loss of generality, here we suppose that the quantum observables have identical spectra of values in both *in*- and *out*- situations. In terms of QM the mentioned supposition means that the operators  $\hat{A}_j$  have the same mathematical expressions in both *in*- and *out*- readings. The respective expressions are the known ones from QM.

In the framework delimited by the above notifications the description of QMS requires putting the transformation (68) in concrete forms and then using some of the known rules of QM. In our opinion the mentioned requirement must be formulated in terms of quantum probabilities carriers. Such carriers are the probabilistic densities  $\rho_\eta$  and currents  $\vec{J}_\eta$  defined by

$$\rho_\eta = |\psi_\eta|^2, \quad \vec{J}_\eta = \frac{\hbar}{m} |\psi_\eta|^2 \cdot \nabla \phi_\eta \quad (69)$$

Here  $|\psi_\eta|$  and  $\phi_\eta$  represents the modulus and the argument of  $\psi_\eta$  respectively (i.e.  $\psi_\eta = |\psi_\eta| \exp(i\phi_\eta)$ ) and  $m$  denotes the mass of microparticle.

The alluded formulation is connected with the observations [84] that the couple  $\rho - \vec{J}$  "encodes the probability distributions of quantum mechanics" and it "is in principle measurable by virtue of its effects on other systems". To be added here the possibility [85] of taking in QM as primary entity the couple  $\rho_{in} - \vec{J}_{in}$  but not the wave function  $\psi_{in}$  (i.e. to start the QM considerations with

the continuity equation for the mentioned couple and subsequently to derive the Schrodinger equation for  $\psi_{in}$ ).

According to the above observations the transformations (68) have to be formulated in terms of  $\rho_\eta$  and  $\vec{J}_\eta$ . But  $\rho_\eta$  and  $\vec{J}_\eta$  refer to the position and the motion kinds of probability respectively. Experimentally the two kinds can be regarded as measurable by distinct devices and procedures. Consequently the mentioned formulation has to combine the following two distinct transformations

$$\rho_{in} \rightarrow \rho_{out}, \quad \vec{J}_{in} \rightarrow \vec{J}_{out} \quad (70)$$

The considerations about the classical relation (62) suggest that, by completely similar arguments, the transformations (70) admit the following transformations

$$\rho_{out}(\vec{r}) = \iiint \Gamma(\vec{r}, \vec{r}') \rho_{in}(\vec{r}') d^3\vec{r}' \quad (71)$$

$$J_{out; \alpha} = \sum_{\beta=1}^3 \iiint \Lambda_{\alpha\beta}(\vec{r}, \vec{r}') J_{in; \beta}(\vec{r}') d^3\vec{r}' \quad (72)$$

In (72)  $J_{\eta; \alpha}$  with  $\eta = in, out$  and  $\alpha = 1, 2, 3 = x, y, z$  denote Cartesian components of  $\vec{J}_\eta$ .

Note the fact that the kernels  $\Gamma$  and  $\Lambda_{\alpha\beta}$  from (71) and (72) have significance of transfer probabilities, completely analogous with the meaning of the classical kernel  $G(a, a')$  from (62). This fact entails the following relations

$$\iiint \Gamma(\vec{r}, \vec{r}') d^3\vec{r}' = \iiint \Gamma(\vec{r}, \vec{r}') d^3\vec{r} = 1 \quad (73)$$

$$\sum_{\alpha=1}^3 \iiint \Lambda_{\alpha\beta}(\vec{r}, \vec{r}') d^3\vec{r}' = \sum_{\beta=1}^3 \iiint \Lambda_{\alpha\beta}(\vec{r}, \vec{r}') d^3\vec{r}' = 1 \quad (74)$$

The kernels  $\Gamma$  and  $\Lambda_{\alpha\beta}$  describe the transformations induced by QMS in the data (information) about the measured system (microparticle). Therefore they incorporate some extra-QM elements regarding the characteristics of measuring devices and procedures. The respective elements do not belong to the usual QM framework which refers to the intrinsic (own) characteristics of the measured system.

The above considerations facilitate an evaluation of the effects induced by QMS on the probabilistic estimators of here considered orbital observables  $A_j$ . Such observables are described by the operators  $\hat{A}_j$  whose expressions depend on  $\vec{r}$  and  $\nabla$ . According to the previous discussions the mentioned operators are supposed to remain invariant under the transformations which describe QMS. So one can say that in the situations associated with the wave functions  $\psi_\eta$  ( $\eta = in, out$ ) the mentioned observables are described by the following probabilistic estimators/characteristics (of first order): mean values  $\langle A_j \rangle_\eta$ , correlations  $C_\eta(A_j, A_k)$  and standard deviations  $\Delta_\eta A_j$ . With the usual notation

$(f, g) = \int f^* g d^3\vec{r}$  for the scalar product of functions  $f$  and  $g$ , the mentioned estimators are defined by the relations

$$\begin{aligned} \langle A_j \rangle_\eta &= \left( \psi_\eta, \hat{A}_j \psi_\eta \right), & \delta_\eta \hat{A}_j &= \hat{A}_j - \langle A_j \rangle_\eta \\ C_\eta(A_j, A_k) &= \left( \delta_\eta \hat{A}_j \psi_\eta, \delta_\eta \hat{A}_k \psi_\eta \right), & \Delta_\eta A_j &= \sqrt{C_\eta(A_j, A_j)} \end{aligned} \quad (75)$$

Add here the fact that the *in* version of the estimators (75) are calculated by means of the wave function  $\psi_{in}$ , known from the considerations about the inner properties of the investigated system (e.g. by solving the corresponding Schrödinger equation). On the other hand the *out* version of the respective estimators can be evaluated by usage of the probability density and current  $\rho_{out}$  and  $\vec{J}_{out}$ . So if  $\hat{A}_j$  does not depend on  $\nabla$  (i.e.  $\hat{A}_j = A_j(\vec{r})$ ) in evaluating the scalar products from (75) one can use the evident equality  $\psi_{out} \hat{A}_j \psi_{out} = \hat{A}_j \rho_{out}$ . When  $\hat{A}_j$  depends on  $\nabla$  (i.e.  $\hat{A}_j = A_j(\nabla)$ ) in the same products can be appealed the substitution

$$\psi_{out}^* \nabla \psi_{out} = \frac{1}{2} \nabla \rho_{out} + \frac{im}{\hbar} \vec{J}_{out} \quad (76)$$

$$\psi_{out}^* \nabla^2 \psi_{out} = \rho_{out}^{\frac{1}{2}} \nabla^2 \rho_{out}^{\frac{1}{2}} + \frac{im}{\hbar} \nabla \vec{J}_{out} - \frac{m^2}{\hbar^2} \frac{\vec{J}_{out}^2}{\rho_{out}} \quad (77)$$

The mentioned usage seems to allow the avoidance of the implications regarding [84] "a possible nonuniqueness of current" (i.e. of the couple  $\rho_\eta - \vec{J}_\eta$ ).

Within the above presented model of QMS the instrumental uncertainty (errors) associated with the measurements of observables  $A_j$  can be evaluated through the following uncertainty indicators

$$\begin{aligned} \varepsilon \{ \langle A_j \rangle \} &= | \langle A_j \rangle_{out} - \langle A_j \rangle_{in} | \\ \varepsilon \{ C(A_j, A_k) \} &= | C_{out}(A_j, A_k) - C_{in}(A_j, A_k) | \\ \varepsilon \{ \Delta A_j \} &= | \Delta_{out} A_j - \Delta_{in} A_j | \end{aligned} \quad (78)$$

These quantum indicators are completely similar with the classical ones (64).

Note that here can be used a similarity with the classical situation discussed in the previous section. So the quantum measuring uncertainties can be evaluated not only by the quantities (78) but also through the changes in informational entropies. In quantum cases the respective entropies can be defined as

$$S(\rho_\eta) = - \iiint \rho_\eta(\vec{r}) \cdot \ln \rho_\eta(\vec{r}) d^3\vec{r} \quad (79)$$

$$S(\vec{J}_\eta) = - \iiint v^{-1} |\vec{J}_\eta(\vec{r})| \cdot \ln |v^{-1} \vec{J}_\eta(\vec{r})| d^3\vec{r} \quad (80)$$

In the last of these relations the factor  $v^{-1}$  was introduced together with  $\vec{J}_\eta$  because of dimensional considerations ( $v$  has the dimension of velocity).

Then in an informational view the measuring uncertainties are described by the couple of the indicators  $\varepsilon \{S(\rho)\}$  and  $\varepsilon \{S(\vec{J})\}$  defined by the following relations :

$$\varepsilon \{S(\rho)\} = |S(\rho_{out}) - S(\rho_{in})| \geq 0 \quad (81)$$

$$\varepsilon \{S(\vec{J})\} = |S(\vec{J}_{out}) - S(\vec{J}_{in})| \geq 0 \quad (82)$$

The relation (81) can be proved similarly with (67). Probably that proof of relation (82) requires more elaborate mathematical reasonings (it was written here on intuitive considerations). In both of the respective relations the sign = refers to the ideal measurements while the cases with > regard the non-ideal measurements.

The here discussed model regarding the description of QMS is exemplified in Annex A.

Now is the place to note that the *out*-version of the estimators (75), (79) and (80) as well as the uncertainty indicators (78), (81) and (82) have a theoretical significance. In practice the verisimilitude of such estimators and indicators must be tested by comparing them with their experimental correspondents (obtained by sampling and processing of the data collected from the recorder of the measuring device). If the test is confirmative both theoretical descriptions, of QM intrinsic properties of system and of QMS, can be considered as adequate. But if the test gives an invalidation of the results, at least one of the mentioned descriptions must be regarded as inadequate.

In the end of this section we wish to add the following two observations:

(i) The here proposed description of QMS does not imply some interconnection of principle between the measuring uncertainties of two distinct observables. This means that from the perspective of the respective description there are no reasons to discuss about a measuring compatibility or incompatibility of two observables.

(ii) The above considerations from the present section refer to the QMS of orbital observables. Similar considerations can be also done in the case of QMS regarding the spin observables. In such a case besides the probabilities of spin-states (well known in QM publications) it is important to take into account the spin current density (e.g. in the version proposed recently [86]).

## 12 Some conclusions

We started the present paper from the observation that in fact CIUR is troubled by a number of still unsolved shortcomings. Then, for a primary goal of our text, we strove to investigate in detail the main aspects as well as the authenticity of the respective shortcomings. So we firstly analysed the renowned deficiencies regarding the pairs of canonically conjugate observables  $L_z - \phi$ ,  $N - \phi$  and  $E - t$ . Additionally we also discussed a whole class of other CIUR shortcomings usually underestimated (or even neglected) in publications.



The mentioned investigations, performed in sections 2 - 6, reveal the following aspects :

- (i) A group of the CIUR's shortcomings appear from the application of the usual (Robertson -Schrodinger) version of UR in situations where, mathematically, it is incorrect ;
- (ii) The rest of the shortcomings result from unnatural linkages with things of other nature (e.g. with the thought experimental relations or with the presence/absence of  $\hbar$  in some formulas);
- (iii) Moreover one finds that, if the mentioned applications and linkages are handled correctly, the alluded shortcomings prove themselves as being veridic and unavoidable facts. The ensemble of the respective facts invalidate all the basic ideas of CIUR.

In consensus with the above noted findings, in Section 7, we promoted the opinion that CIUR must be abandoned as an incorrect and useless (or even misleading) doctrine. Conjointly with the respective opinion we think that the primitive UR (the so called Heisenberg's relations) must be regarded as:

- (i) fluctuation formulas - in their theoretical (Robertson-Schrodinger) version,
- (ii) fictitious things, without any physical significance - in their thought-experimental version.

Because CIUR ideas imply suppositions regarding QMS the above announced abandonment requires a re-examination ( at least in part) of the QMS problems. To such a requirement we tried to answer in Sections 8 - 11. So, by a detailed investigation, we have shown that the CIUR-connected approaches of QMS are grounded on dubitable (or even incorrect) premises. That is why we declare ourselves for reconsidered approaches of QMS, based on new (and more natural) premises. Such an approach is argued and developed in Sections 9, 10 and 11 respectively, and it is exemplified in Annex A.

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## Annex A: An exemplification

For a simple exemplification of the model presented in Section 11 let us refer to a microparticle in a one-dimensional motion along the  $x$ -axis. We take  $\psi_{in}(x) =$

$|\psi_{in}(x)| \cdot \exp\{i\phi_{in}(x)\}$  with

$$|\psi_{in}(x)| = \left(\sigma\sqrt{2\pi}\right)^{-\frac{1}{2}} \cdot \exp\left\{-\frac{(x-x_0)^2}{4\sigma^2}\right\}, \quad \phi_{in}(x) = kx \quad (83)$$

Correspondingly we have

$$\rho_{in}(x) = |\psi_{in}(x)|^2, \quad J_{in}(x) = \frac{\hbar k}{m} |\psi_{in}(x)|^2 \quad (84)$$

So the intrinsic properties of the microparticle are described by the parameters  $x_0$ ,  $\sigma$  and  $k$ .

If the errors induced by QMS are small the kernels  $\Gamma$  and  $\Lambda$  in (71)-(72) can be considered of Gaussian forms like

$$\Gamma(x, x') = \left(\gamma\sqrt{2\pi}\right)^{-1} \cdot \exp\left\{-\frac{(x-x')^2}{2\gamma^2}\right\} \quad (85)$$

$$\Lambda(x, x') = \left(\lambda\sqrt{2\pi}\right)^{-1} \cdot \exp\left\{-\frac{(x-x')^2}{2\lambda^2}\right\} \quad (86)$$

where  $\gamma$  and  $\lambda$  describe the characteristics of the measuring devices. Then for  $\rho_{out}$  and  $J_{out}$  one finds

$$\rho_{out}(x) = [2\pi(\sigma^2 + \gamma^2)]^{-\frac{1}{2}} \cdot \exp\left\{-\frac{(x-x')^2}{2(\sigma^2 + \gamma^2)}\right\} \quad (87)$$

$$J_{out}(x) = \hbar k [2\pi m^2(\sigma^2 + \lambda^2)]^{-\frac{1}{2}} \cdot \exp\left\{-\frac{(x-x')^2}{2(\sigma^2 + \lambda^2)}\right\} \quad (88)$$

It can be seen that in the case when both  $\gamma \rightarrow 0$  and  $\lambda \rightarrow 0$  the kernels  $\Gamma(x, x')$  and  $\Lambda(x, x')$  degenerate into the Dirac's function  $\delta(x-x')$ . Then  $\rho_{out} \rightarrow \rho_{in}$  and  $J_{out} \rightarrow J_{in}$ . Such a case corresponds to an ideal measurement. Alternatively the cases when  $\gamma \neq 0$  and/or  $\lambda \neq 0$  are associated with non-ideal measurements.

As observables of interest we take coordinate  $x$  and momentum  $p$  described by the operators  $\hat{x} = x \cdot$  and  $\hat{p} = -i\hbar \frac{\partial}{\partial x}$ . Then according to the scheme presented in Section 11 one obtains

$$\langle x \rangle_{in} = \langle x \rangle_{out} = x_0, \quad \langle p \rangle_{in} = \langle p \rangle_{out} = \hbar k \quad (89)$$

$$C_{in}(x, p) = C_{out}(x, p) = \frac{i\hbar}{2} \quad (90)$$

$$\Delta_{in}x = \sigma, \quad \Delta_{out}x = \sqrt{\sigma^2 + \gamma^2} \quad (91)$$

$$\Delta_{in}p = \frac{\hbar}{2\sigma} \quad (92)$$

$$\Delta_{out}p = \hbar \sqrt{\frac{k^2(\sigma^2 + \gamma^2)}{\sqrt{(\sigma^2 + \lambda^2)(\sigma^2 + 2\gamma^2 - \lambda^2)}} - k^2 + \frac{1}{4(\sigma^2 + \gamma^2)}} \quad (93)$$

Subsequently, according to (78), the measuring errors regarding  $x$  and  $p$  are characterized by the uncertainty indicators

$$\varepsilon\{\langle x \rangle\} = 0, \quad \varepsilon\{\langle p \rangle\} = 0, \quad \varepsilon\{C(x, p)\} = 0 \quad (94)$$

$$\varepsilon\{\Delta x\} = \sqrt{\sigma^2 + \gamma^2} - \sigma \quad (95)$$

$$\varepsilon\{\Delta p\} = |\Delta_{out}p - \Delta_{in}p| \quad (96)$$

(Read the last formula by appealing to (92) and (93)).

The relations (94)-(96) show that in the considered model the characteristics  $\langle x \rangle$ ,  $\langle p \rangle$  and  $C(x, p)$  of stochastic observables  $x$  and  $p$  are not troubled by measuring errors. But in the same model the characteristics  $\Delta x$  and  $\Delta p$  are disturbed by non-null such errors.

For the measuring uncertainties (81) and (82) regarding the informational entropies one finds

$$\varepsilon\{S(\rho)\} = S(\rho_{out}) - S(\rho_{in}) = \frac{1}{2} \ln \left( 1 + \frac{\gamma^2}{\sigma^2} \right) \quad (97)$$

$$\varepsilon\{S(J)\} = S(J_{out}) - S(J_{in}) = \frac{1}{2} \ln \left( 1 + \frac{\lambda^2}{\sigma^2} \right) \quad (98)$$

In the last of these relation for  $v$  introduced in (80) we take  $v = (m/\hbar k)$ .

For an evaluation of the interdependence between the uncertainty indicators of  $x$  and  $p$  from (94)-(96) one obtains

$$\varepsilon\{\langle x \rangle\} \cdot \varepsilon\{\langle p \rangle\} = 0 \quad (99)$$

$$\varepsilon\{\Delta x\} \cdot \varepsilon\{\Delta p\} \geq \hbar\mu \quad (100)$$

Here  $\mu$  is a real, non-negative and dimensionless quantity which can be evaluated by means of the relations (94)-(96).

If in (83) we restrict to the values  $x_0 = 0$ ,  $k = 0$  and  $\sigma = \sqrt{\frac{\hbar}{2m\omega}}$  our system is just a linear oscillator in its ground state ( $m =$  mass and  $\omega =$  angular frequency). As observable of interest we consider the energy described by the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{m\omega^2}{2} x^2 \quad (101)$$

Then for the respective observable one finds

$$\langle H \rangle_{in} = \frac{\hbar\omega}{2}, \quad \Delta_{in}H = 0 \quad (102)$$

$$\langle H \rangle_{out} = \frac{\omega \left[ \hbar^2 + (\hbar + 2m\omega\gamma^2)^2 \right]}{4(\hbar + 2m\omega\gamma^2)} \quad (103)$$

$$\Delta_{out} H = \frac{\sqrt{2} m \omega^2 \gamma^2 (\hbar + m\omega\gamma^2)}{(\hbar + 2m\omega\gamma^2)} \quad (104)$$

The corresponding instrumental errors are described by the uncertainty indicators

$$\varepsilon \{ \langle H \rangle \} = | \langle H \rangle_{out} - \langle H \rangle_{in} | \neq 0 \quad (105)$$

$$\varepsilon \{ \Delta H \} = | \Delta_{out} H - \Delta_{in} H | \neq 0 \quad (106)$$

## List of abbreviations

**BI** = basic ideas  
 CIUR = conventional interpretation of uncertainty relations  
**CP** = conventional premises  
 EXR = extended rotations  
*in* = input  
**NP** = new premises  
*out* = output  
 QM = quantum mechanics  
 QMS = quantum measurements  
 QTP = quantum torsion pendulum  
**R** = remark  
 SRC = sharp circular rotations  
*srtc* = super-resolution-thought-experimental  
*te* = thought-experimental  
 UR = uncertainty relation(s)

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