Is Electromagnetic Field Momentum Due to the Flow of Field Energy?*

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Abstract

Momentum and energy conservation require electromagnetic field momentum and energy to be treated as physically real, even in static fields. This motivates the conjecture that field momentum might be due to the flow of a relativistic mass density (defined as energy density divided by the square of the speed of light).

This article investigates the velocity of such a mass flow and finds a conflict between two different definitions of it, both of which originally seem plausible if the flow is to be taken as real. This investigation is careful to respect the transformation rules of special relativity throughout.

The paper demonstrates that the consensus definition of the flow velocity of electromagnetic energy is inconsistent with the transformation rules of special relativity, and hence is incorrect. A replacement flow velocity is derived which is completely consistent with those transformation rules.

The conclusion is that these conflicting definitions of flow velocity cannot be resolved in a way that is consistent with special relativity and also allows electromagnetic field momentum density to be the result of relativistic mass flow. Though real, field momentum density cannot be explained as the flow of a relativistic mass density.

As a byproduct of the study, it is also shown that there is a comoving system in which the electromagnetic energy-momentum tensor is reduced to a simple diagonal form, with two of its diagonal elements equal to the energy density and the other two diagonal elements equal to plus and minus a single parameter derived from the electromagnetic field values, a result that places constraints on possible fluid models of electromagnetism.

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1 Introduction

Explanation of electromagnetic field momentum as due to the flow of field energy depends crucially on a *correct* definition for the velocity of that energy flow. (Since special relativity is the invariance theory of electromagnetism, throughout this paper by *correct* or *valid* we will mean that a definition or construct is correct or valid only if it is consistent with special relativity.)

In the modern, post-relativity era it has been the consensus in the literature, at least since the first English edition of Born and Wolf's Optics,[1] that the energy flow velocity is $\mathbf{V}_{\mathrm{A}} = \mathbf{S}/\mathcal{E} = \mathbf{G}/\mathcal{M}_{\mathrm{rel}}$ where $\mathbf{S}$ is the Poynting vector, $\mathcal{E}$ is the energy density, $\mathbf{G}$ is the momentum density, and $\mathcal{M}_{\mathrm{rel}} = \mathcal{E}/c^2$ is the so-called relativistic mass density. We refer to this as definition A. With this definition, multiplying through gives $\mathbf{G} = \mathcal{M}_{\mathrm{rel}}\mathbf{V}_{\mathrm{A}}$ which would exhibit momentum density as due to flow of relativistic mass. This article investigates this consensus claim and finds reason to doubt it. Sections 2 and 3 demonstrate that the coordinate flow velocity definition $\mathbf{V}_{\mathrm{A}}$ is not *correct* in the above sense. It is not consistent with the transformation rules of special relativity.

The example of a rotating disk with a magnet at its center and charged spheres on its perimeter provides a convincing argument that, to preserve the principle of angular momentum conservation, the field momentum

[1] In a discussion of the Poynting theorem in material media, but with no special attention to Lorentz covariance, Born and Wolf [3] Section 14.2, eq.(8) identify $\mathbf{V}_{\mathrm{A}}$ as the *velocity of energy transport* or *ray velocity.* (The first edition of Born and Wolf's text appeared in 1959.) Section B.2 of Smith [20] echos Born and Wolf but provides no new derivation. Geppert [7] makes the same identification. More recently, Sebens [18, 19] relies on these and other sources to identify $\mathbf{V}_{\mathrm{A}}$ as the EM mass flow velocity. Sebens also considers earlier, pre-Einstein studies by Poincare. The present paper however is focussed on the reconsideration of the subject forced by special relativity.

of even a static electromagnetic (EM) field must be considered physically real.[2] It is also generally assumed that conservation of energy requires the energy density of the EM field to be physically real, even for static fields. The issue here is not the reality of the field momentum, but only the question of its explanation as due to the flow of relativistic mass.

The velocity of the energy (mass) flow at a given event can also be defined as the velocity of an observer who measures the Poynting energy flux vector to be zero at that event. If $\mathbf{S}$ truly is the flux of energy flow, then an observer comoving with this flow should observe a zero value of that energy flux. This is definition B and its velocity will be denoted as $\mathbf{V}_{\rm B}$. Its details are presented in Section 4 and its caveats in Section 5.

Section 6 demonstrates that the comoving reference system used in the derivation of definition B allows a reduction of the EM energy-momentum tensor to a simple, diagonal form, with two of its diagonal elements equal to the energy density in the comoving frame and the other two diagonal elements equal to plus and minus a single parameter derived from the EM field values. This reduction of the EM energy-momentum tensor is shown to place important constraints on fluid-dynamic models of energy flow in the EM field.

Section 7 concludes that EM field momentum density cannot be explained as the flow of an EM mass density. Since $\mathbf{V}_{\rm A} \neq \mathbf{V}_{\rm B}$, a choice between them must be made. But neither choice leads to a relativistically correct model in which field momentum density is due to the motion of a relativistic mass density.

Section 8 accepts the negative results of the present study of relativistic mass flow, and speculates about possible ways forward in the search for a model, if any, of physics that might underly the Maxwell Equations.[3]

[2] Feynman et al [6], Section 17-4, Section 27-6, and Figure 17-5. Quantitative matches of field to mechanical angular momentum are found, for example, in Romer [17] and Boos [2].

[3] Some material from this paper was previously published in preliminary and incomplete form in Johns[10]. However, all such material is reorganized and extensively expanded. Also, the present paper adds the proof in Section 3 that the definition of $\mathbf{V}_{\rm A}$ violates the transformation rules of special relativity and hence is invalid. An added Afterword in Section 8 suggests reasons for the failure of field momentum to be the flow of field energy. This section also speculates on the consequences of this failure for future attempts to find new physics beneath the Maxwell equations.

This paper uses Heaviside-Lorentz units. We denote four-vectors as $\mathsf{K} = K^0\mathsf{e}_0 + \mathbf{K}$ where e_0 is the time unit vector and the three-vector part is understood to be $\mathbf{K} = K^1\mathsf{e}_1 + K^2\mathsf{e}_2 + K^3\mathsf{e}_3$. In the Einstein summation convention, Greek indices range from 0 to 3, Roman indices from 1 to 3. The Minkowski metric tensor is $(\eta_{\alpha\beta}) = (\eta^{\alpha\beta}) = \mathrm{diag}(-1,+1,+1,+1)$. Three-vectors are written with bold type $\mathbf{K}$, and their magnitudes as K. Thus $|\mathbf{K}| = K$.

2 Detail of Definition A

Definition A of the energy flow velocity is defined by two equivalent[4] formulas

$$\mathbf{S} = \mathcal{E}\mathbf{V}_{\rm A} \qquad \text{and} \qquad \mathbf{V}_{\rm A} = \mathbf{S}/\mathcal{E} \tag{1}$$

Writing $\mathbf{S}{=}c\,(\mathbf{E}\times\mathbf{B})$ and $\mathcal{E}{=}\left(E^2+B^2\right)/2$, in terms of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, gives[5]

$$\mathbf{V}_{\rm A} = \frac{\mathbf{S}}{\mathcal{E}} = \frac{2c\,(\mathbf{E}\times\mathbf{B})}{(E^2+B^2)} \tag{2}$$

It is easily shown from the inequalities $(E-B)^2 \geq 0$ and $EB \geq |\mathbf{E}\times\mathbf{B}|$ that $|\mathbf{V}_{\rm A}| \leq c$.

The consensus definition that energy flow velocity is simply energy flux density divided by energy density is suggested by analogy with the well understood example of $\mathbf{V}_{\rm q} = \mathbf{J}/\rho$ as the velocity of charge flow, given electric charge density ρ and charge flux density $\mathbf{J}$. The analogy suggests $\mathbf{V}_{\rm A} = \mathbf{S}/\mathcal{E}$ as the velocity of electromagnetic energy, given energy density $\mathcal{E}$ and energy flux density $\mathbf{S}$. It will therefore be useful to begin with a review of the properties of charge flow.

The divergence of the EM field four-tensor is the charge flux four-vector $\mathsf{J} = c\rho\mathsf{e}_0 + \mathbf{J}$. This four-vector can be timelike, spacelike, or null. But there are useful cases (*e.g.,* flow of particles all having the same sign of electric charge) in which it is timelike. In these cases, the velocity $\mathbf{V}_{\rm q} = \mathbf{J}/\rho$ has magnitude less than the speed of light and so one can write $\mathbf{J} = \rho\mathbf{V}_{\rm q}$, in analogy to $\mathbf{S} = \mathcal{E}\mathbf{V}_{\rm A}$ above.

The four-vector flux can then be written as

$$\mathsf{J} = \rho\left(c\mathsf{e}_0 + \mathbf{V}_{\rm q}\right) = (\rho/\gamma_{\rm q})\,\mathsf{U}_{\rm q} \tag{3}$$

[4] Since $\mathcal{E}$ is nonzero except when $E{=}B{=}0$, throughout this paper we will take definition A, $\mathbf{V}_{\rm A} = \mathbf{S}/\mathcal{E}$, to be equivalent to the formula $\mathbf{S} = \mathcal{E}\mathbf{V}_{\rm A}$.

[5] Electromagnetic formulas in this paper are taken from Griffiths [8] and Jackson [9], with translation into Heaviside-Lorentz units.

where $\mathbf{U}_q = \gamma_q(c\mathbf{e}_0 + \mathbf{V}_q)$ is the four-vector velocity corresponding to $\mathbf{V}_q$, and $\gamma_q = \left\{1 - V_q^2/c^2\right\}^{-1/2}$.

Since $\mathbf{J}$ is known to be a four-vector, a Lorentz boost transformation[6] with the boost velocity $\mathbf{V} = \mathbf{V}_q$ transforms from the original unprimed reference system to a reference system (which we call a comoving system and denote by a prime) in which $\mathbf{J}' = 0$.[7]

The next section of the paper will show that the analogy between $\mathbf{V}_q = \mathbf{J}/\rho$ and $\mathbf{V}_A = \mathbf{S}/\mathcal{E}$ is flawed and that the latter formula is inconsistent with the transformation rules of special relativity.

In addition to the above analogy with charge density, the following simple geometric construction can be used to argue for the consensus definition $\mathbf{V}_A = \mathbf{S}/\mathcal{E}$.

> *Geometry of a Flow:* Given a flowing substance with density κ and flux density vector $\mathbf{K}$, define a velocity as $\mathbf{v} = \mathbf{K}/\kappa$.
>
> Now we must examine this velocity definition $\mathbf{v}$ to see whether it passes the test of compatibility with the rules of special relativity. If it does not, then application of the results of this inset will be a misapplication, and would lead to results inconsistent with the special theory of relativity. The test must be applied on a case-by-case basis. Some applications will be seen to be correct, but others will be misapplications.
>
> Assuming that $\mathbf{v}$ passes that test, the following simple geometric argument may be made. Consider an arbitrarily oriented area element $d\mathbf{a}$ and a time increment dt. The product $d\tau = (\mathbf{v}\,dt)\cdot d\mathbf{a}$ is a volume element. All points in $d\tau$ moving with velocity $\mathbf{v}$ will flow through $d\mathbf{a}$ in time dt. Now multiply by κ to obtain $\kappa d\tau = \kappa\mathbf{v}\cdot d\mathbf{a}\,dt$, the amount of substance in $d\tau$. If we assume that all of the substance in $d\tau$ is moving with the same velocity $\mathbf{v}$, then $\mathbf{v}\cdot d\mathbf{a}\,dt$ is the amount of substance flowing through $d\mathbf{a}$ in time dt. But this amount is also, by definition of the flux density $\mathbf{K}$, given by $\mathbf{K}\cdot d\mathbf{a}\,dt$. Thus
>
> $$\mathbf{K}\cdot d\mathbf{a}\,dt = \kappa\mathbf{v}\cdot d\mathbf{a}\,dt \qquad (4)$$
>
> Since $d\mathbf{a}$ and dt are arbitrary, it follows that $\mathbf{K} = \kappa\mathbf{v}$.
>
> But the assumption that all elements of the substance are moving with the same velocity $\mathbf{v}$ is often unjustified. (Physically plausible, timelike charge flows, especially ones with charges of different sign, will in general have many different charge velocities.) Then the above simple geometric argument fails. Its *assumption of equal velocities* is violated by physically plausible cases, and so the simple argument cannot be used to prove in general that $\mathbf{K} = \kappa\mathbf{v}$.
>
> But if the argument in eq.(4) fails (while still assuming that $\mathbf{v}$ passes the relativity test above) we can consider $\mathbf{v} = \mathbf{K}/\kappa$ to be a *definition* of an average flow velocity. Then $\mathbf{K} = \kappa\mathbf{v}$ is true by definition.
>
> Note that κ and $\mathbf{K}$ are the *fundamental* quantities that appear in consevation equations like $\nabla\cdot\mathbf{K} + \partial\kappa/\partial t = 0$. Velocities like $\mathbf{v} = \mathbf{K}/\kappa$ are *derived* quantities that must pass the test of consistency with special relativity. When they pass, they allow the conservation equation to be simplified by substituting $\mathbf{K} = \kappa\mathbf{v}$. But this simplification is not essential. The conservation equation itself is still valid even when $\mathbf{v} = \mathbf{K}/\kappa$ fails to pass the relativity test and so cannot be used.

If the application with $\kappa = \mathcal{E}$, $\mathbf{K} = \mathbf{S}$ passed the test of compatibility with relativity, it would predict that the energy flow velocity must be $\mathbf{v} = \mathbf{S}/\mathcal{E} = \mathbf{V}_A$, the consensus definition of energy flow velocity defined in eq.(2).

The above two arguments for the consensus definition of energy flow velocity are examined in the following section.

3 Caveats of Definition A

The difficulty with Definition A is that it is inconsistent with the transformation rules of special relativity.

[6] The Lorentz boost formalism is summarized in Appendices I and I.1.

[7] See Appendix II for proof.

We take a flow velocity definition $\mathbf{v}$ to be *relativistically valid* only if that definition passes a simple test using the Einstein velocity addition formula. Since that test is derived directly from the transformation rules of special relativity, any flow velocity definition that fails the test must also violate some rule of special relativity. Definition A fails this simple test and hence is not relativistically valid.

> *Einstein Addition Test:* Consider two alternate reference systems referred to as the unprimed and asterisk systems, and let the asterisk system be obtained from the unprimed one by a boost transformation with boost velocity $\mathbf{V} = V\mathbf{e}_1$ of arbitrary magnitude V. Then $\mathbf{e}_i$ are parallel to the corresponding $\mathbf{e}^*_i$, for $i = 1, 2, 3$. Let the velocity of a comoving observer moving with the energy flow be $\mathbf{v}$ when derived from the unprimed system, and the velocity of that same observer when derived from the asterisk system be $\mathbf{v}^*$. With no loss of generality, the asterisk system can be oriented so that $\mathbf{v}^*$ is in the $\mathbf{e}^*_1$ direction. Then $\mathbf{v}^* = v^*\mathbf{e}^*_1$ and $\mathbf{v} = v\mathbf{e}_1$.
>
> Electrodynamics in vacuum except for a possible explicit source four-vector J can be expressed in manifestly covariant form and therefore must be true in any reference system; there can be no privileged reference system. So any relativistically correct derivation that defines flow velocity $\mathbf{v}$ when applied in the unprimed system can also be applied to define flow velocity definition $\mathbf{v}^*$ when applied in the asterisk system. And these two velocities are velocities of the same comoving observer. Thus, when we make a boost transformation between the unprimed and asterisk systems, these velocities must transform (See Example 12.6 of Griffiths[8]) as
>
> $$(v/c) = \frac{(V/c) + (v^*/c)}{1 + (V\,v^*/c^2)} \tag{5}$$
>
> This is the Einstein addition test that any relativistically valid flow velocity definition must pass.
>
> Note that this test is a *necessary condition* for consistency with the transformation rules of special relativity. Any electromagnetic energy flow definition that fails to pass the *Einstein Addition Test* in eq.(5) must necessarily violate the transformation rules of special relativity from which eq.(5) is directly derived.

We first consider the argument from analogy between $\mathbf{V}_{\rm A} = \mathbf{S}/\mathcal{E}$ and $\mathbf{V}_{\rm q} = \mathbf{J}/\rho$. The choice $\mathbf{V}_{\rm A} = \mathbf{S}/\mathcal{E}$ in definition A is based in part on the analogy with $\mathbf{V}_{\rm q} = \mathbf{J}/\rho$ for electric charge flow. However, that analogy has a crucial limitation.

It is known that $\mathsf{J} = c\rho\mathsf{e}_0 + \mathbf{J}$ is a legitimate four-vector whose components transform according to the standard rule $J'^{\alpha} = \Lambda^{\alpha}_{\ \beta} J^{\beta}$ from Appendix II. But it is also known that the analogous expression $\mathsf{S} \asymp c\mathcal{E}\mathsf{e}_0 + \mathbf{S}$ is *not* a four-vector. (I use the symbol $\asymp$ to remind the reader that, though written formally as a four-vector here, it does not actually transform as one.) There is no such four-vector S. Writing $S^0 = c\mathcal{E}$ and $S^i = (\mathbf{S})_i$, these components do *not* transform according to the four-vector rule.[8] In this case, $S'^{\alpha} \neq \Lambda^{\alpha}_{\ \beta} S^{\beta}$. Rather, noting that $\mathbf{S} = c^2\mathbf{G}$ where $\mathbf{G}$ is the linear momentum density of the EM field, the $c\mathcal{E}$ and $(\mathbf{S})_i$ actually transform as the (00) and (0i) components of a four-tensor $cT^{\alpha\beta}$ defined as c times the standard EM energy-momentum tensor

$$\left(T^{\alpha\beta}\right) = \begin{pmatrix} \mathcal{E} & c\,G_1 & c\,G_2 & c\,G_3 \\ c\,G_1 & M_{11} & M_{12} & M_{13} \\ c\,G_2 & M_{21} & M_{22} & M_{23} \\ c\,G_3 & M_{31} & M_{32} & M_{33} \end{pmatrix} \tag{6}$$

where $M_{ij} = -\left(E_iE_j + B_iB_j\right) + \frac{1}{2}\,\delta_{ij}\left(E^2 + B^2\right)$. Thus the transformation rule for $S^0 = c\mathcal{E}$ and $S^i = (\mathbf{S})_i$ is as the (00) and (01) components of the more complicated expression

$$cT'^{\alpha\beta} = \Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu} cT^{\mu\nu} \tag{7}$$

which will also involve contributions from the cM_{ij} terms.

This failure of S actually to be a four-vector has the consequence that the velocity $\mathbf{V}_{\rm A} = \mathbf{S}/\mathcal{E}$ does *not* pass the

[8] A related point is made by Rohrlich [16], using the so-called von Laue's theorem to argue that *integrals* of $c\mathcal{E}$ and $\mathbf{S}$ over hyperplanes may in some cases transform as four-vectors. But, we are treating these quantities locally, at a particular event. Von Laue's theorem does not imply that the local field functions $c\mathcal{E}$ and $\mathbf{S}$ (the integrands of these hyperplane integrals) themselves transform as components of a four-vector. They do not. See also Chapter 6 of Rohrlich [15].

above test, and hence is not a relativistically valid flow velocity definition. It does not obey the transformation rules of special relativity from which the above Einstein velocity test is derived. To see this failure, begin with the example of the charge flow definition $\mathbf{V}_{\mathrm{q}} = \mathbf{J}/\rho$ that *does* pass the test.

Appendix III demonstrates that the coordinate velocities $\mathbf{V}_{\mathrm{q}} = \mathbf{J}/\rho$ and $\mathbf{V}^*_{\mathrm{q}} = \mathbf{J}^*_{\mathrm{q}}/\rho^*$ derived from the charge density four-vector **J** *do* pass the *Einstein Addition Test,* as must be true for any well-defined coordinate velocity. When the inverse four-vector transformation rule $J^\alpha = \underline{\Lambda}^\alpha{}_\beta J^{*\beta}$ is used to write J^0 and J^i in terms of **V** and the asterisk system quantities J^{*0} and J^{*i}, the result is the last expression in eq.(42), which agrees with the Einstein velocity addition rule eq.(5)and hence with special relativity.

However, if we now attempt to apply this same argument to the case of $\mathbf{V}_{\mathrm{A}} = \mathbf{S}/\mathcal{E}$ and $\mathbf{V}^*_{\mathrm{A}} = \mathbf{S}^*/\mathcal{E}^*$, the argument of Appendix III fails. In this case, the *equality* $J^\alpha = \underline{\Lambda}^\alpha{}_\beta J^{*\beta}$ is replaced by the *inequality* $S^\alpha \neq \underline{\Lambda}^\alpha{}_\beta S^{*\beta}$ resulting from the failure of **S** to be a four-vector. Thus, eqs.(41, 42) are not true when **J** is replaced by **S**, and the argument does not go through to its conclusion.

To write $\mathbf{V}_{\mathrm{A}} = \mathbf{S}/\mathcal{E}$ in terms of **V** and the asterisk system quantities requires that we use the transformation rule

$$cT^{\alpha\beta} = \underline{\Lambda}^\alpha{}_\mu \underline{\Lambda}^\beta{}_\nu cT^{*\mu\nu} \tag{8}$$

derived from eq.(7). The quantities $c\mathcal{E}$ and and S_i are the (00) and (0i) components of four-tensor $cT^{\alpha\beta}$ rather than components of a four-vector as in the **J** case. As a result, when written in terms of asterisk quantities using eq.(8), the quantities $\mathcal{E}c$ and **S**, and hence their ratio $\mathbf{V}_{\mathrm{A}}/c = \mathbf{S}/\mathcal{E}c$, are more complex formulas and contain cM^*_{ij} terms. In place of the equality in eq.(42) for the charge density case, in the case of $\mathbf{V}_{\mathrm{A}}$ we have the *inequality*

$$(V_{\mathrm{A}}/c) = (S/\mathcal{E}c) \neq \frac{(V/c) + \left(V^*_{\mathrm{A}}/c\right)}{1 + \left(V\,V^*_{\mathrm{A}}/c^2\right)} \tag{9}$$

where the expression on the extreme right in eq.(9) is the correct Einstein velocity addition result.[9] Thus $\mathbf{V}_{\mathrm{A}}$ fails the *Einstein Addition Test.*

[9] Compare eq.(12.3) of Griffiths[8] and the formula on the extreme right in eq.(42).

The inequality in eq.(9) can also be derived directly from the transformation rules for the E and B fields, without making any reference to the charge flow results. Using the same geometry as in the *Einstein Addition Test* above, and a transformation rule similar to eq.(12), it can be shown that

$$(V_{\mathrm{A}}/c) = \frac{(V/c) + \left(1 + V^2/c^2\right)\left(V^*_{\mathrm{A}}/c\right)}{(1 + V^2/c^2) + 2\left(V\,V^*_{\mathrm{A}}/c^2\right)} \neq \frac{(V/c) + \left(V^*_{\mathrm{A}}/c\right)}{1 + \left(V\,V^*_{\mathrm{A}}/c^2\right)} \tag{10}$$

which corroborates the inequality in eq.(9) and shows again that V_{A} fails the *Einstein Addition Test.*

Thus, unlike the velocity definition $\mathbf{V}_{\mathrm{q}} = \mathbf{J}/\rho$, the definition $\mathbf{V}_{\mathrm{A}} = \mathbf{S}/\mathcal{E}$ does not pass the *Einstein Addition Test,* and therefore is not a valid coordinate velocity definition in the sense of being consistent with the transformation rules of special relativity. The analogy between charge flow and electromagnetic energy flow fails to prove that $\mathbf{V}_{\mathrm{A}} = \mathbf{S}/\mathcal{E}$ is a relativistically valid definition.

We now consider the argument in *Geometry of a Flow* in Section 2 for the consensus definition V_{A} for the energy flow velocity. Application of *Geometry of a Flow* to a case is equivalent to asserting that there is a velocity $\mathbf{v} = \mathbf{K}/\kappa$ such that $\mathbf{K} = \kappa\mathbf{v}$. In a study such as this paper that demands strict agreement with special relativity (as must any theory of the Maxwell equations), we must also investigate whether the velocity **v** is in agreement with the rules of relativity theory. When it does agree, the application of *Geometry of a Flow* is the correct application. If it does not, then it is a misapplication that implies a relativistically incorrect result.

An example of a correct application is the flow of electric charge with a timelike 4-vector **J**. In this case, Appendix III shows that the velocity $\mathbf{V}_{\mathrm{q}} = \mathbf{J}/\rho$ passes the *Einstein Addition Test* and thus agrees with relativity theory; thus this application is correct.

However, caution is required. The 4-vector **J** can also be spacelike (superpose two timelike flows in opposite directions with charge density of nearly equal magnitude but opposite sign). In this case the velocity $\mathbf{v} = \mathbf{J}/\rho$ will be greater than the speed of light which is forbidden by relativity theory. The application of the *Geometry of a Flow* to this case is therefore a misapplication. Its application results in a velocity $\mathbf{v} = \mathbf{J}/\rho$ that violates a rule of special relativity. In the case of spacelike **J**, there is no relativistically correct velocity $\mathbf{v}$ such that $\mathbf{J} = \rho\mathbf{v}$.

Application of *Geometry of a Flow* to the case of electromagnetic energy flow is also a misapplication. As seen in eqs.(9 and 10), it predicts a velocity $\mathbf{v} = \mathbf{S}/\mathcal{E} = V_A$ that fails the *Einstein Addition Test* and is thus inconsistent with the transformation rules of special relativity. There is no relativistically correct velocity $\mathbf{v}$ such that $\mathbf{S} = \mathcal{E}\mathbf{v}$. The following corollary is implied:

> *Corollary 1:* The equivalent formulas
>
> $$\mathbf{S} = \mathcal{E}\mathbf{v} \qquad \text{and} \qquad \mathbf{v} = \mathbf{S}/\mathcal{E} \tag{11}$$
>
> cannot be used to derive a relativistically valid definition $\mathbf{v}$ for the electromagnetic energy flow velocity. There is no relativistically valid flow velocity definition $\mathbf{v}$ which satisfies eq.(11).
> *Proof:* The equation eq.(11) has only the single, unique solution $\mathbf{v} = V_A$, and this solution is relativistically invalid. That an equation has only an invalid solution implies that the equation has no valid solutions, which is Corollary 1.

In summary, regardless of how it is derived, either from a flawed analogy with charge flow, or with a misapplication of *Geometry of a Flow,* the definition $\mathbf{V}_A = \mathbf{S}/\mathcal{E}$ of energy flow velocity violates the transformation rules of special relativity and is not relativistically valid. It follows that $\mathbf{V}_A$ cannot be the velocity of electromagnetic energy flow in a relativistically correct theory.

4 Detail of Definition B

Definition B of the energy flow velocity, denoted $\mathbf{V}_B$, is the velocity of a comoving observer who measures a zero energy flux. Expressed in the precise language of Lorentz boost transformations:

> The coordinate velocity of the flow of electromagnetic field energy at a given event is the velocity $\mathbf{V}_B$ of a Lorentz boost that transforms the original reference system into a reference system in which the Poynting energy flux vector is zero at that event.

An observer at that event and at rest in this transformed system, which we call the comoving system and denote by primes, therefore measures a zero energy flux. The zero flux measurement indicates that this observer is comoving with the flow of energy. Such an observer has coordinate velocity $\mathbf{V}_B$ relative to the original system,[10] and therefore $\mathbf{V}_B$ is the coordinate velocity of the energy flow at the given event.

The problem is to find this boost velocity $\mathbf{V}_B$. An analogous problem arises in the generic theory of relativistic fluid flow.[11] There a velocity can be defined as $\mathbf{V}_a = \mathbf{p}c^2/e$ analogous to our $\mathbf{V}_A = \mathbf{S}/\mathcal{E} = \mathbf{G}c^2/\mathcal{E}$. But, a proof analogous to the proof in Section 3 shows that velocity to be inconsistent with the Einstein velocity relation of special relativity and hence not a valid definition. In the theory of fluid flow, there is no other way to derive a flow velocity from first principles. One solution is simply to *assert* that there *must be* a primed reference system moving with the flow even though we have been unable to derive it, and to assert that the energy-momentum tensor in that system must have the isotropic form $X'^{\alpha\beta} = \text{diag}\{\varepsilon, \pi, \pi, \pi\}$. This is called the *perfect fluid* model. However it remains true that the flow velocity and the form of the energy-momentum tensor are simply asserted rather than derived.[12]

In the electromagnetic case considered in the present paper, however, the failure of $\mathbf{V}_A$ does not exhaust our ways of deriving $\mathbf{V}_B$. We can fall back on the rich structure of the Maxwell equations themselves, which underlie the definition of the energy-momentum tensor $T^{\alpha\beta}$ and from which it was derived. Thus in the electromagnetic case we are not reduced to merely *asserting* the existence of a comoving frame. We can actually *derive* the boost velocity $\mathbf{V}_B$ and the form of the energy-momentum tensor in the comoving frame, starting from first principles.

The rules for transformation of electric and magnetic fields by a boost with velocity $\mathbf{V}_B$ can be written in a special relativistically correct but not manifestly covariant

[10] See Appendix I.2 for a demonstration that any point at rest in the primed system moves with coordinate velocity $\mathbf{V}_B$.

[11] Part I, Chapter 2 of Weinberg[21] presents what I will refer to as a *generic* theory. It assumes only that a fluid is composed of a countable set of small particles characterized by their mass m_n, position $\mathbf{x}_n$, and velocity $\mathbf{v}_n$. Weinberg (*e.g.* his eq.(2.8.1) *et seq*) uses the language of Dirac delta function densities, but his formulas are easily translated into more standard density functions.

[12] Weinberg[21] Part I, Chapter 2, Section 10, eq.(2.10.1) *et seq*. Note that Weinberg introduces the perfect fluid by saying, "A useful approximation is ..." rather than attempting to derive it from his previous work in his Chapter 2.

form[13]

$$\mathbf{E}' \doteq \gamma_B\left(\mathbf{E} + \frac{\mathbf{V}_B}{c}\times\mathbf{B}\right) + (1-\gamma_B)\,\frac{\mathbf{V}_B\,(\mathbf{V}_B\cdot\mathbf{E})}{V_B^2}$$
$$\mathbf{B}' \doteq \gamma_B\left(\mathbf{B} - \frac{\mathbf{V}_B}{c}\times\mathbf{E}\right) + (1-\gamma_B)\,\frac{\mathbf{V}_B\,(\mathbf{V}_B\cdot\mathbf{B})}{V_B^2} \qquad (12)$$

where the Lorentz factor is $\gamma_B = \left(1 - V_B^2/c^2\right)^{-1/2}$.

The boost velocity $\mathbf{V}_B$ can then be found by writing

$$\mathbf{V}_B = \lambda\,\mathbf{V}_A \qquad (13)$$

where λ is a rotationally scalar quantity to be determined. The velocity $\mathbf{V}_B$ will have the same *direction* as $\mathbf{V}_A$ but not the same *magnitude.*

Since $\mathbf{V}_A$ and hence $\mathbf{V}_B$ are perpendicular to both the electric and magnetic fields, it follows that $(\mathbf{V}_B\cdot\mathbf{E}) = (\mathbf{V}_B\cdot\mathbf{B}) = 0$. Thus, eq.(12) reduces to[14]

$$\mathbf{E}' \doteq \gamma_B\left(\mathbf{E} + \frac{\mathbf{V}_B}{c}\times\mathbf{B}\right)$$
$$\mathbf{B}' \doteq \gamma_B\left(\mathbf{B} - \frac{\mathbf{V}_B}{c}\times\mathbf{E}\right) \qquad (14)$$

Insert eq.(14) into the equation for the Poynting vector in the comoving system, $\mathbf{S}' = c\mathbf{E}'\times\mathbf{B}'$. Using property (a) of the symbol $\doteq$ from footnote 13 together with eq.(13) and then eq.(2) leads to[15]

$$\mathbf{S}' = c\mathbf{E}'\times\mathbf{B}' \doteq \gamma_B^2 c\,(\mathbf{E}\times\mathbf{B})\left((V_A/c)^2\,\lambda^2 - 2\lambda + 1\right) \qquad (15)$$

Choose λ to solve the quadratic equation

$$\left((V_A/c)^2\lambda^2 - 2\lambda + 1\right) = 0 \qquad (16)$$

Then eq.(15) makes $\mathbf{S}' = 0$, as required by definition B. The solution is

$$\lambda = \frac{1}{(V_A/c)^2}\left\{1 - \sqrt{1-(V_A/c)^2}\right\} \qquad (17)$$

From eq.(13), the definition B for the velocity of the energy flow is therefore

$$\mathbf{V}_B = \lambda\,\mathbf{V}_A = \frac{1}{(V_A/c)^2}\left\{1 - \sqrt{1-(V_A/c)^2}\right\}\mathbf{V}_A \qquad (18)$$

where $\mathbf{V}_A$ is defined in eq.(2).

This $\mathbf{V}_B$ is the relativistically correct boost velocity from the original unprimed frame to the comoving reference frame in which $\mathbf{S}' = 0$.[16]

Since $\mathbf{V}_B$ is parallel to the energy flux vector $\mathbf{S}$, the energy flow velocity can also be written as $\mathbf{V}_B = V_B\,(\mathbf{S}/S)$ where the magnitude V_B is given by[17]

$$(V_B/c) = \frac{1}{(V_A/c)}\left\{1 - \sqrt{1-(V_A/c)^2}\right\} \qquad (19)$$

Eq.(19) can be inverted to give

$$(V_A/c) = \frac{2\,(V_B/c)}{1+(V_B/c)^2} \qquad (20)$$

which can be used to write the factor λ in eq.(17) as a function of the velocity definition B

$$\lambda = \frac{1+(V_B/c)^2}{2} \qquad (21)$$

Summary: This section uses the co-variant field transformation equations in eq.(12) to derive a boost velocity, $\mathbf{V}_B$, defined in eq.(18), that transforms from the unprimed system to a comoving primed system in which the energy flux vector S'=0. Then Appendix I.2 shows that $\mathbf{V}_B$ is also the coordinate velocity relative to the unprimed system of an observer at rest in the comoving primed system. Since it is derived directly from the rules of special relativity, this velocity is well defined and relativistically correct. Also, it can be shown that this $\mathbf{V}_B$ passes the Einstein Addition Test, as it must.

The observer at rest in the primed comoving system will observe the energy flux vector $\mathbf{S}'$

[13] See Section 11.10 of Jackson [9], eq.(11.149). The $\doteq$ symbol means that the components of the three-vector on the left side of this symbol, expressed in the primed coordinate system, are numerically equal to the corresponding components of the three-vector on the right side of this symbol, expressed in the original unprimed system. If $\mathbf{a}' \doteq \mathbf{c}$ and $\mathbf{b}' \doteq \mathbf{d}$, it is easily proved that: (a) $(\mathbf{a}'\times\mathbf{b}') \doteq (\mathbf{c}\times\mathbf{d})$ and (b) $(\mathbf{a}'\cdot\mathbf{b}')=(\mathbf{c}\cdot\mathbf{d})$. (c) Also if $\mathbf{w}' \doteq \mathbf{w}$ then the magnitudes are equal, $w' = |\mathbf{w}'| = |\mathbf{w}| = w$.

[14] Note that $\mathbf{V}_B' \doteq \mathbf{V}_B$ as defined in Appendix I.1, together with eq.(14) and property (b) of the symbol $\doteq$ in footnote 13, imply that $(\mathbf{V}_B'\cdot\mathbf{E}')=\mathbf{V}_B\cdot\gamma\,[\mathbf{E} + (\mathbf{V}_B/c)\times\mathbf{B}] = \gamma(\mathbf{V}_B\cdot\mathbf{E})=0$. Similarly, $(\mathbf{V}_B'\cdot\mathbf{B}')=0$.

[15] See a detailed derivation of eq.(15) in Appendix IV.

[16] Appendix V gives details of the comoving system for possible values of $(\mathbf{E}\cdot\mathbf{B})$ at a given event.

[17] The text just after eq.(2) proves that $0 \leq V_A \leq c$. As (V_A/c) increases from 0 to 1, eq.(19) shows that (V_B/c) increases monotonically from 0 to 1, with $V_B \leq V_A$ at every point. It follows that $0 \leq V_B \leq c$ also. Regions of the unprimed system where $\mathcal{E}$ is nonzero but $\mathbf{S}$ is zero have $V_A = 0$ and $V_B = 0$, and have no energy flow.

to be zero. Thus if he holds an oriented area element $d\mathbf{a}'$ in any orientation he will find that the energy flux through that element to be $\mathbf{S}' \cdot d\mathbf{a}' = 0$. Hence the observer must be moving at the same velocity as the flow of energy, and its velocity will be the same as his velocity, $\mathbf{V}_{\mathrm{B}}$.

The conclusion is that the well defined and relativistically correct coordinate velocity $\mathbf{V}_{\mathrm{B}}$ must be the correct velocity of the electromagnetic energy flow.

This conclusion, together with $\mathbf{V}_{\mathrm{B}} \neq \mathbf{V}_{\mathrm{A}}$ from eq.(18), also gives independent conformation of the results of Section 3, that the definition of electromagnetic energy flow velocity is not the consensus value $\mathbf{V}_{\mathrm{A}}$. It is important to note that this conclusion, along with all the results in Section 4, depends only on the assumption of the standard transformation laws of electromagnetic field in eq.(12), and not on any other assumptions.

Thus Section 4 provides convincing proof that $\mathbf{V}_{\mathrm{B}}$ is the relativistically correct electromagnetic energy flow velocity definition, and that the consensus value $\mathbf{V}_{\mathrm{A}}$ is not.

5 Caveats of Definition B

The caveats for definition A are technical; they concern its violation of the transformation rules of special relativity. By contrast, the derivation of $\mathbf{V}_{\mathrm{B}}$ in Section 4 is completely consistent with special relativity throughout. Velocity $\mathbf{V}_{\mathrm{B}}$ is the relativistically correct velocity of an observer at rest in the primed comoving reference system, defined as a system in which the energy flux vector $\mathbf{S}' = 0$. It follows that $\mathbf{V}_{\mathrm{B}}$ is the relativistically valid energy flow velocity.

But one may question whether the condition $\mathbf{S}' = 0$ truly implies that the comoving observer is moving at the same velocity as the underlying energy flow, as required for $\mathbf{V}_{\mathrm{B}}$ to be the correct energy flow velocity. For, as eq.(2) and eq.(18) directly prove, $\mathbf{V}_{\mathrm{B}} \neq \mathbf{V}_{\mathrm{A}} = \mathbf{S}/\mathcal{E}$ and hence $\mathbf{S} \neq \mathcal{E}\mathbf{V}_{\mathrm{B}}$. Since $\mathbf{V}_{\mathrm{B}}$ is relativistically valid, Corollary 1 also proves equality to be impossible in that equation. It may seem that this inequality will block derivation of the Poynting theorem on which the meanings of $\mathbf{S}$ and $\mathcal{E}$ depend.

However, the equality of $\mathbf{S}$ and $\mathcal{E}\mathbf{V}_{\mathrm{B}}$ is *not* a necessary condition for the Poynting theorem, or for $\mathbf{V}_{\mathrm{B}}$ to be the correct flow velocity. Derivation of the Poynting theorem is independent of the relation between $\mathbf{S}$ and the product $\mathcal{E}\mathbf{V}_{\mathrm{B}}$. The Poynting conservation of energy theorem derives from the divergence of the symmetric energy-momentum tensor $T^{\mu\nu}$ defined in eq.(6)

$$\partial_\mu T^{\mu\nu} = -f^\nu \quad \text{where} \quad f^\alpha = \frac{1}{c} F^\alpha{}_\beta J^\beta \tag{22}$$

is the Lorentz force four-vector and $F^{\mu\nu}$ is the electromagnetic field tensor.[18] The $\nu = 0$ component of the above manifestly covariant equation expands to

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{J} \tag{23}$$

which is the Poynting work-energy theorem of electromagnetism. Since it is derived from the manifestly covariant pair of equations, eq.(22) the Poynting energy conservation formula eq.(23) is well defined and relativistically correct. And the meanings of $\mathcal{E}$ and $\mathbf{S}$ as energy density and energy-flux vector, respectively, are established by eq.(23). No further proof is required. The Poynting theorem and the meaning of $\mathbf{S}$ as the energy flux vector are thus proved, regardless of the relation between $\mathbf{S}$ and the product $\mathcal{E}\mathbf{V}_{\mathrm{B}}$.

This proof that the Poynting theorem and the meaning of the energy flux vector $\mathbf{S}$ are independently established corroborates and completes the Summary at the end of Section 4, which depended on the meaning of $\mathbf{S}$. Thus we are driven to the conclusion that $\mathbf{V}_{\mathrm{B}}$ is indeed the well defined and relativistically correct velocity of the electromagnetic energy flow, and that $\mathbf{V}_{\mathrm{A}}$ is not.

Another caveat to Definition B might be the formula $\mathbf{S} \neq \mathcal{E}\mathbf{V}_{\mathrm{B}}$ itself. It might seem necessary that $\mathbf{S} = \mathcal{E}\mathbf{V}_{\mathrm{B}}$. But the definition eq.(2) shows $\mathbf{S} = \mathcal{E}\mathbf{V}_{\mathrm{B}}$ to be equivalent to $\mathbf{V}_{\mathrm{B}} = \mathbf{S}/\mathcal{E} = \mathbf{V}_{\mathrm{A}}$, whereas eq.(18) proves that $\mathbf{V}_{\mathrm{B}} \neq \mathbf{V}_{\mathrm{A}}$. Thus $\mathbf{S} = \mathcal{E}\mathbf{V}_{\mathrm{B}}$ is untrue. Also, the suggestion that $\mathbf{S} = \mathcal{E}\mathbf{V}_{\mathrm{B}}$ is based on a flawed assumption. Note that the quantities $\mathbf{S}$ and $\mathcal{E}$ from the Poynting theorem are *densities, not precise values.* Thus, like all densities, they are *averages,* sums of underlying quantities divided by an averaging volume. Thus the underlying velocities appearing in the definition of density $\mathbf{S}$ contain a range

[18] See Section 7.3 of Rindler[14].

of velocity values, only averaging to $\mathbf{S}$. But if we, unjustifiably and contrary to the absence of precise values, were to assume that all of these velocities were equal to $\mathbf{V}_B$, then the simple geometric argument in *Geometry of a Flow* in Section 2 would establish $\mathbf{S} = \mathcal{E}\mathbf{V}_B$. But this assumption of equal velocities is unjustified and contrary to the fact that $\mathbf{S}$ and $\mathcal{E}$ are densities and not exact values. Thus this assumption is flawed, and its conclusion cannot be accepted. Therefore, $\mathbf{S} \neq \mathcal{E}\mathbf{V}_B$ is the correct result.

Although $\mathbf{V}_A \neq \mathbf{V}_B$ in general, there is an important exceptional case which the theory here must approach as a limit. A plane, monochromatic, right /left circularly polarized light wave in vacuum with angular velocity ω and wave vector $\mathbf{k} = (\omega/c)\,\mathbf{e}_3$ has

$$\begin{aligned} \mathbf{E} &= E_0\,\{\mathbf{e}_1\ \cos\phi \pm \mathbf{e}_2\ \sin\phi\} \\ \mathbf{B} &= E_0\,\{\mp\mathbf{e}_1\ \sin\phi + \mathbf{e}_2\ \cos\phi\} \end{aligned} \tag{24}$$

where $\phi = (kz - \omega t)$ and $z = x^3$. This EM field has $\mathbf{E} \perp \mathbf{B}$ and $E = B = E_0 \neq 0$, which is the limiting case treated in item (c) of Appendix V. In this exceptional case, velocity definitions A and B coincide. As can be seen from eq.(2) and eq.(19) $\mathbf{V}_B = \mathbf{V}_A$ and $V_B = V_A = c$.

As noted in Appendix V, and as also can be read from eq.(27), in this case $\mathcal{E}'$ would be zero in the comoving system. But there is no comoving system with velocity magnitude equal to the speed of light. Observers are not permitted to ride on light waves. However, both definitions do agree that the flow speed of a light wave is the speed of light.

Setting $V_A = c$ and using $\mathbf{S} = c^2\mathbf{G}$, eq.(2) in this special case implies that

$$Gc = \mathcal{E} \tag{25}$$

Since wave solution eq.(24) defines a mode of the EM field whose second-quantization creates photons of definite vector momentum, eq.(25) can be considered a classical precursor of the relation $pc = e$ for the photon momentum and energy, a relation that requires the photon to be a massless particle.

6 The Energy-Momentum Tensor in a Comoving Frame

The derivation of velocity $\mathbf{V}_B$ in Section 4 also allows the electromagnetic energy-momentum tensor in the comoving system to be derived from first principles. As noted in Section 4, the comoving energy-momentum tensor of a perfect fluid must simply be asserted rather than derived. But the electromagnetic energy-momentum tensor in a comoving system can be *derived,* and shown equal to a simple, diagonal form depending only on the energy density and one other parameter.

In the comoving (primed) coordinate system that was produced by the Lorentz boost $\mathbf{V}_B$, the energy-momentum tensor eq.(6) is represented by the tensor components $T'^{\alpha\beta}$ in which the $cG'_i = S'_i/c = 0$.

$$\left(T'^{\alpha\beta}\right) = \begin{pmatrix} \mathcal{E}' & 0 & 0 & 0 \\ 0 & M'_{11} & M'_{12} & M'_{13} \\ 0 & M'_{21} & M'_{22} & M'_{23} \\ 0 & M'_{31} & M'_{32} & M'_{33} \end{pmatrix} \tag{26}$$

where[19]

$$\mathcal{E}' = \frac{1}{2}\left(E'^2 + B'^2\right) = \mathcal{E}\,\frac{1 - (V_B/c)^2}{1 + (V_B/c)^2} \tag{27}$$

$$\text{and} \quad M'_{ij} = -\left(E'_i E'_j + B'_i B'_j\right) + \delta_{ij}\mathcal{E}'$$

We can now make another Lorentz transformation, an orthogonal spatial rotation at fixed time, to diagonalize the real, symmetric sub-matrix M' in eq.(26).

The required spatial rotation can be defined as the product of two proper rotations. First rotate the coordinate system to bring the $\mathbf{e}'_3$ axis into the $\mathbf{V}'_B \doteq \mathbf{V}_B$ direction.[20] Denote this rotated system by tildes. Rotations do not change three-vectors, which are invariant objects under rotations. However, rotations do change the *components* of three-vectors. Thus $\tilde{\mathbf{V}}_B = \mathbf{V}'_B$, $\tilde{\mathbf{E}} = \mathbf{E}'$, and $\tilde{\mathbf{B}} = \mathbf{B}'$, but in the tilde system $\tilde{\mathbf{V}}_B$ now has components $\tilde{V}_{B1} = \tilde{V}_{B2} = 0$ and $\tilde{V}_{B3} = V_B$. Then using footnote 14 on page 7, we have $0 = (\mathbf{E}' \cdot \mathbf{V}'_B) = (\tilde{\mathbf{E}} \cdot \tilde{\mathbf{V}}_B) = V_B\tilde{E}_3$. Except in no-flow regions with $\mathcal{E}$ nonzero but $\mathbf{S}$ zero, the magnitude $V_B \neq 0$ and thus $\tilde{E}_3 = 0$. A similar argument proves that $\tilde{B}_3 = 0$. Thus the (33) component of the energy-momentum tensor when expressed in the tilde system is $\tilde{T}^{33} = -\left(\tilde{E}_3^2 + \tilde{B}_3^2\right) + \tilde{\mathcal{E}} = \tilde{\mathcal{E}}$. The tensor from eq.(26), when expressed in the tilde system, becomes

$$\left(\tilde{T}^{\alpha\beta}\right) = \begin{pmatrix} \tilde{\mathcal{E}} & 0 & 0 & 0 \\ 0 & \tilde{M}_{11} & \tilde{M}_{12} & 0 \\ 0 & \tilde{M}_{21} & \tilde{M}_{22} & 0 \\ 0 & 0 & 0 & \tilde{\mathcal{E}} \end{pmatrix} \tag{28}$$

[19] Eq.(27) is derived in Appendix IV.

[20] Note that item (c) of footnote 13 implies equal magnitudes $V'_B = V_B$.

where $\tilde{\mathcal{E}} = \mathcal{E}'$.

Since the invariant trace of the electrodynamic energy-momentum tensor vanishes,[21] it follows from eq.(28) that

$$0 = \eta_{\alpha\beta}\tilde{T}^{\alpha\beta} = -\tilde{\mathcal{E}} + \tilde{M}_{11} + \tilde{M}_{22} + \tilde{\mathcal{E}} \tag{29}$$

and hence $\tilde{M}_{11} = -\tilde{M}_{22}$. Also, the symmetry of the energy-momentum tensor makes $\tilde{M}_{21} = \tilde{M}_{12}$. Thus

$$\left(\tilde{T}^{\alpha\beta}\right) = \begin{pmatrix} \tilde{\mathcal{E}} & 0 & 0 & 0 \\ 0 & -\tilde{\psi} & \tilde{\xi} & 0 \\ 0 & \tilde{\xi} & \tilde{\psi} & 0 \\ 0 & 0 & 0 & \tilde{\mathcal{E}} \end{pmatrix} \tag{30}$$

where $\tilde{\psi} = \tilde{M}_{22}$ and $\tilde{\xi} = \tilde{M}_{12}$.

A second proper rotation, this time about the $\tilde{\mathbf{e}}_3$ axis, produces the final coordinate system, denoted with double primes. After this rotation, $E''_3 = \tilde{E}_3 = 0$, $B''_3 = \tilde{B}_3 = 0$, and $\mathbf{V}''_{\mathrm{B}} = \tilde{\mathbf{V}}_{\mathrm{B}}$ has components $V''_{\mathrm{B}1} = V''_{\mathrm{B}2} = 0$ and $V''_{\mathrm{B}3} = V_{\mathrm{B}}$. The only effect of this second rotation is to diagonalize the 2x2 matrix $\begin{pmatrix} -\tilde{\psi} & \tilde{\xi} \\ \tilde{\xi} & \tilde{\psi} \end{pmatrix}$. The energy-momentum tensor then has its final, diagonal form in the double-prime system

$$T''^{\alpha\beta} = \begin{pmatrix} \mathcal{E}'' & 0 & 0 & 0 \\ 0 & -a'' & 0 & 0 \\ 0 & 0 & a'' & 0 \\ 0 & 0 & 0 & \mathcal{E}'' \end{pmatrix} \tag{31}$$

where $\mathcal{E}'' = \tilde{\mathcal{E}} = \mathcal{E}'$. The parameter a'' has absolute value $|a''| = \left\{\tilde{\psi}^2 + \tilde{\xi}^2\right\}^{1/2}$ where $\pm\left\{\tilde{\psi}^2 + \tilde{\xi}^2\right\}^{1/2}$ are the two eigenvalues of the matrix $\begin{pmatrix} -\tilde{\psi} & \tilde{\xi} \\ \tilde{\xi} & \tilde{\psi} \end{pmatrix}$ that were calculated during the diagonalization process. The sign of a'' depends on the directions and relative magnitudes of the electric and magnetic fields.

The rotation that takes the system from the primed to the double-primed system is then the product of the first and second rotations. The various representations of the boost velocity used above are related by $\mathbf{V}''_{\mathrm{B}} = V_{\mathrm{B}}\mathbf{e}''_3 = \tilde{\mathbf{V}}_{\mathrm{B}} = V_{\mathrm{B}}\tilde{\mathbf{e}}_3 = \mathbf{V}'_{\mathrm{B}} \doteq \mathbf{V}_{\mathrm{B}}$. It follows from item (c) of footnote 13 that all of these vectors have the same original magnitude V_{B}.

The energy-momentum tensor eq.(31) in the double-prime system is diagonal and in a canonical form, with two elements equal to $\mathcal{E}'' = \mathcal{E}'$ and two other elements equal to plus or minus the single parameter a''.

The reduction of the EM energy-momentum tensor to the diagonal form in eq.(31) has important consequences for possible fluid-dynamic models of EM energy flow. For example, the perfect fluid model[22] has a comoving energy-momentum tensor given by the diagonal matrix $\left(X^{\alpha\beta}\right) = \mathrm{diag}\,(e, p, p, p)$ where e is the energy density and the p are isotropic pressure terms, all of which are equal by definition. But, regardless of the value of parameters $\mathcal{E}''$ and a'', there is no choice of e and p for which the quadruplet of numbers (e, p, p, p) can match the quadruplet of numbers $(\mathcal{E}'', -a'', a'', \mathcal{E}'')$, other than the unphysical case when all of the numbers in both quadruplets are zero. Similarly, the so-called dust model[23] has $\left(X^{\alpha\beta}\right) = \mathrm{diag}\,(e, 0, 0, 0)$ which also cannot match the EM tensor.

7 Conclusion

Since $\mathbf{V}_{\mathrm{A}} \neq \mathbf{V}_{\mathrm{B}}$, it is necessary to make a choice between definitions A and B.

First, consider the definition A. This definition preserves the idea that EM field momentum density is due to the flow of a relativistic mass. Dividing eq.(1) by c^2 and using $\mathbf{G} = \mathbf{S}/c^2$ and $\mathcal{M}_{\mathrm{rel}} = \mathcal{E}/c^2$ gives

$$\mathbf{G} = \mathcal{M}_{\mathrm{rel}}\mathbf{V}_{\mathrm{A}} \tag{32}$$

But it preserves that idea at the cost of inconsistency with the transformation rules of special relativity. As shown in Section 3, $\mathbf{V}_{\mathrm{A}}$ is not a legitimate coordinate velocity three-vector.

Also, since $\mathbf{V}_{\mathrm{A}} \neq \mathbf{V}_{\mathrm{B}}$, definition A is not consistent with the seemingly inescapable condition that $\mathbf{S}'$ must vanish in the comoving system, a condition which should be encompassed by any acceptable definition of flow velocity.

Definition B, on the other hand, is derived with complete adherence to the transformation rules of special relativity. But it does not preserve the identification of field momentum density with moving relativistic mass.

[21] See Section 7.8 of Rindler [14]

[22] See Part I, Chapter 2, Section 10, eq.(2.10.1) *et seq* of Weinberg [21].

[23] Discussed in Section 12.2 of d'Inverno [4] and on page 301 *et seq.* of Rindler [14].

Introducing the factor λ from eq.(13) into eq.(32) gives

$$\mathbf{G} = \frac{\mathcal{M}_{\rm rel}}{\lambda}(\lambda \mathbf{V}_A) = \mathcal{M}_{\rm eff}\mathbf{V}_B \tag{33}$$

where $\mathcal{M}_{\rm eff} = \mathcal{M}_{\rm rel}/\lambda$ is an effective mass density appropriate to definition B.

Using eq.(21), this effective mass is seen to be

$$\mathcal{M}_{\rm eff} = \frac{\mathcal{M}_{\rm rel}}{\lambda} = \frac{2\mathcal{M}_{\rm rel}}{1 + (V_{\rm B}/c)^2} \tag{34}$$

which is larger than $\mathcal{M}_{\rm rel}$. Eq.(34) thus gives a quantitative measure of the inadequacy of relativistic mass flow as an explanation of EM field momentum density. Flow of relativistic mass $\mathcal{M}_{\rm rel}$ at velocity $\mathbf{V}_B$ would produce a momentum density that has the same direction as $\mathbf{G}$ but has a magnitude that is too small by the factor $\lambda \leq 1$ defined in eqs.(17, 21).

Note that this failure of the flow of relativistic mass $\mathcal{M}_{\rm rel}$ to explain the field momentum density $\mathbf{G}$ in the EM fields *must not be confused with the so-called hidden momentum* in the sources that is sometimes invoked to balance the field momentum and preserve momentum conservation globally.[24]

The present paper is concerned only with a correct understanding of the EM *field contribution itself,* locally at every point of the EM field including those points with no source density. Encouraged by the arguments from the Feynman example noted in footnote 2 above, we accept that the vector $\mathbf{G} = \mathbf{S}/c^2$ correctly reproduces the local field momentum density at every point of the EM field. The question is the source of that local point-by-point field momentum density.

Individual consideration of the two velocities $\mathbf{V}_{\rm A}$ and $\mathbf{V}_{\rm B}$ above has concluded that the flow of relativistic mass cannot provide a relativistically correct explanation of the local field momentum. The same conclusion can be reached more directly using Corollary 1 of Section 3:

> *Corollary 2:* There is no relativistically correct energy flow velocity definition $\mathbf{v}$ that satisfies the equation
>
> $$\mathbf{G} = \mathcal{M}_{\rm rel}\mathbf{v} \tag{35}$$
>
> *Proof:* To prove Corollary 2, assume Corollary 1. Divide eq.(12) by c^2 and use $\mathbf{G} = \mathbf{S}/c^2$ and $\mathcal{M}_{\rm rel} = \mathcal{E}/c^2$. The result is eq.(35). Thus Corollary 1 implies Corollary 2.

The conclusion is that there is no relativistically correct definition of mass flow velocity that explains the EM field momentum density as the flow of a relativistic mass density. The title question of the paper has a negative answer. When adherence to the strict transformation rules of special relativity is required, electromagnetic field momentum cannot be explained as due to the flow of field energy.

8 Afterword

Detailed studies of the energy and momentum carried by the electromagnetic field, such as the present paper, can be seen as searches for clues to a possible new physics underlying the Maxwell Equations. But, if we accept the conclusion at the end of Section 7, the attempt to model the Maxwell Equations at the level of energy/mass flow and the energy-momentum tensor seems a failed program.

This failure calls into question the whole project of finding a deeper level behind the Maxwell Equations. A consensus exists that the electric and magnetic fields are not states of anything else,[25] but are either abstract mathematical aids, or themselves elements of reality to be taken as fundamental. In this view, the Maxwell Equations are already at the fundamental level, and attempts to derive them from some deeper reality are a futile revival of nineteenth century aether theories and, as Feynman says, "... produce nothing but errors."

But Maxwell himself looked for fluid models of his equations. Maxwell [12] explains the inverse square electric force law as a consequence of the spread of an incompressible fluid. And he later proposes (Maxwell [11]) a model of Faraday's magnetic field lines based on fluid vortices.[26]

Perhaps, instead of taking the conclusion in Section 7 as a reason to abandon Maxwell's search, we should rather read a lesson from it: *Our attempt at a flow model may have failed because the attempt is taking place at the wrong level.* The electromagnetic energy-momentum tensor $T^{\alpha\beta}$ is quadratic in the fundamental

[24] See Example 12.12 of Griffiths [8], and also McDonald [13] and Babson et al [1].

[25] See, for example, Section 4-5 of Feynman [6].

[26] Falconer [5] surveys other early vortex models.

electromagnetic fields E and B. It may be that any successful flow model of electrodynamics must operate at the linear, field level and not at the energy-momentum level.

For example, consider two monochromatic plane waves propagating in the $+\mathbf{e}_3$ direction, wave a with right circular polarization and wave b with left circular polarization.

$$\mathbf{E}_a = E_0 \{\mathbf{e}_1 \cos\phi + \mathbf{e}_2 \sin\phi\}$$
$$\mathbf{E}_b = E_0 \{\mathbf{e}_1 \cos\phi - \mathbf{e}_2 \sin\phi\} \quad (36)$$

where $\phi = (kz - \omega t)$and $z = x^3$, and the magnetic fields are the cross product of $\mathbf{e}_3$ with the given electric fields. The electromagnetic energy-momentum tensors $T_a^{\alpha\beta}$ and $T_b^{\alpha\beta}$ of these two waves are the same, with

$$\left(T_a^{\alpha\beta}\right) = \left(T_b^{\alpha\beta}\right) = \begin{pmatrix} E_0^2 & 0 & 0 & E_0^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_0^2 & 0 & 0 & E_0^2 \end{pmatrix} \quad (37)$$

We now want to superpose these two situations a and b. The superposition of the two circularly polarized waves is a linearly polarized wave

$$\mathbf{E}_{a+b} = \mathbf{E}_a + \mathbf{E}_b = 2E_0\mathbf{e}_1 \cos\phi \quad (38)$$

and the resulting energy-momentum tensor is

$$\left(T_{a+b}^{\alpha\beta}\right) = \begin{pmatrix} 4E_0^2\cos^2\phi & 0 & 0 & 4E_0^2\cos^2\phi \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4E_0^2\cos^2\phi & 0 & 0 & 4E_0^2\cos^2\phi \end{pmatrix} \quad (39)$$

which is time varying at each fixed spatial point, passing through zero every π/ω seconds.

This example illustrates that representing the electromagnetic energy-momentum flow as the flow of a fluid at the quadratic energy-momentum level ignores the fact that electromagnetism is a linear theory with superposition. It is difficult to see how combining the two tensors in eq.(37) could result in the time-varying tensor of eq.(39). Electromagnetic fields do not superpose at the energy-momentum level. Therefore an attempt to model electromagnetism at that level is bound to fail. Such a model should be applied at the linear, field level of the E and B fields themselves.

But, in spite of the appeal and long history of Maxwell's quest, there are formidable hurdles facing any model at the field level, even using modern mathematical techniques. One such hurdle is that a complete model of the E and B fields would probably also need to include interaction with, and characterization of, the source fields ρ and $\mathbf{J}$. And it should include not only the effects of sources on fields but also the effects of fields on sources, the Lorentz force law.

Appendix I: Lorentz Boosts

Consider a Lorentz transformation from an "unprimed" coordinate system with coordinates $x = (x^0, x^1, x^2, x^3)$ to a "primed" coordinate system with coordinates $x' = (x'^0, x'^1, x'^2, x'^3)$ where $x^0 = ct$ and $x'^0 = ct'$. The most general proper, homogeneous Lorentz transformation from the unprimed to the primed systems can be written as a Lorentz boost times a rotation.[27]

I.1: Definition of Lorentz Boost

A Lorentz boost transformation is parameterized by a boost velocity vector $\mathbf{V}$ with components (V_1, V_2, V_3) and magnitude $V = \left(V_1^2 + V_2^2 + V_3^2\right)^{1/2}$. Using the Einstein summation convention, it is written as $x'^\alpha = \Lambda^\alpha{}_\beta\, x^\beta$ where $\Lambda^0{}_0 = \gamma$, $\Lambda^0{}_i = \Lambda^i{}_0 = -\gamma V_i/c$, and $\Lambda^i{}_j = \delta_{ij} + (\gamma - 1)V_i V_j/V^2$. The δ_{ij} is the Kroeneker delta function, and $\gamma = \left(1 - V^2/c^2\right)^{-1/2}$.

The inverse boost $\underline{\Lambda}^\alpha{}_\beta$ is the same except for the substitution $V_i \to -V_i$. Thus the inverse boost vector is $(-\mathbf{V}')$ where $\mathbf{V}' \doteq \mathbf{V}$. (See footnote 13 for definition of the $\doteq$ symbol.)

I.2: Meaning of the Boost Velocity V

The velocity $\mathbf{V}$ that parameterizes the Lorentz boost is also the coordinate velocity, as measured from the unprimed system, of any point that is at rest in the primed system. In this sense, the entire primed system is moving with velocity $\mathbf{V}$ as observed from the unprimed system.

To see this, apply the inverse Lorentz boost to the differentials of a point at rest in the primed system, $dx'^i = 0$ for $i = 1, 2, 3$, but $dx'^0 > 0$. The result is $dx^0 = \gamma dx'^0$ and $dx^i = \gamma (V_i/c)\, dx'^0$. Thus $dx^i/dt = V_i$, as was asserted.

[27] See Part I, Chapter 2, Section 1 of Weinberg [21].

Appendix II: Proof that Boost with $\mathbf{V_q}$ Makes $\mathbf{J}' = 0$

As applied to a four-vector $\mathbf{J} = J^0\mathbf{e}_0 + \mathbf{J}$, with $J^0 = c\rho$ and $J^i = (\mathbf{J})_i$ the Lorentz boost transformation rule is $J'^{\alpha}=\Lambda^{\alpha}{}_{\beta}\, J^{\beta}$. Hence

$$\begin{aligned} J'^i &= \Lambda^i{}_0 J^0 + \Lambda^i{}_j J^j \\ &= -\gamma \frac{V_i}{c} J^0 + J^i + (\gamma - 1)\frac{V_i\left(V_j J^j\right)}{V^2} \end{aligned} \quad (40)$$

Replacing boost velocity ratio V_i/c by $\left(\mathbf{V}_{\mathrm{q}}\right)_i/c = J^i/J^0$ in eq.(40) makes $J'^i = 0$, as asserted.

Appendix III: Proof that $\mathbf{V_q}$ is Consistent with the Einstein Velocity Addition Formula

Consider two reference frames, one denoted as unprimed and the other with asterisks. Let the asterisk frame orientation be chosen so that in it $\mathbf{V}_{\mathrm{q}}^* = V_{\mathrm{q}}^*\mathbf{e}_1^*$. Consider the asterisk frame to be obtained from the unprimed frame by a boost velocity $\mathbf{V} = V\mathbf{e}_1$ with the axis directions of the two frames coinciding.

The four-vector charge flux in the asterisk system is then $\mathbf{J} = J^{*0}\mathbf{e}_0^* + J^{*1}\mathbf{e}_1^*$ where $J^{*0} = c\rho^*$ and $J^{*1} = (\mathbf{J}^*)_1$. From the standard inverse boost formula $J^{\alpha} = \underline{\Lambda}^{\alpha}{}_{\beta}\, J^{*\beta}$, the transform between the two frames is (suppressing the 2 and 3 components for simplicity)

$$\begin{pmatrix} J^0 \\ J^1 \end{pmatrix} = \gamma \begin{pmatrix} 1 & (V/c) \\ (V/c) & 1 \end{pmatrix} \begin{pmatrix} J^{*0} \\ J^{*1} \end{pmatrix} \quad (41)$$

The velocities V_{q} and V_{q}^* in the two frames are therefore related by

$$\begin{aligned} \frac{V_{\mathrm{q}}}{c} &= \frac{J^1}{J^0} = \frac{(V/c)\, J^{*0} + J^{*1}}{J^{*0} + (V/c)\, J^{*1}} \\ &= \frac{(V/c) + \left(J^{*1}/J^{*0}\right)}{1 + (V/c)\left(J^{*1}/J^{*0}\right)} = \frac{(V/c) + \left(V_{\mathrm{q}}^*/c\right)}{1 + \left(V\, V_{\mathrm{q}}^*/c^2\right)} \end{aligned} \quad (42)$$

which replicates the standard Einstein velocity addition formula, as asserted. Comparison of eq.(42) and eq.(5) shows that V_{q} passes the *Einstein Velocity Test.*

Appendix IV: Detailed Derivations of Eq.(15) and Eq.(27).

To derive eq.(15), we have eq.(2), eq.(13), eq.(14), and $(\mathbf{V}\cdot\mathbf{E}) = (\mathbf{V}\cdot\mathbf{B}) = 0$. Using eq.(14),

$$\mathbf{S}' = c\,(\mathbf{E}' \times \mathbf{B}') \doteq c\gamma_{\mathrm{B}}^2 \left\{(\mathbf{E}\times\mathbf{B}) + \mathbf{f} + \mathbf{g}\right\}$$

where, omitting zero terms,

$$\begin{aligned} \mathbf{f} &= -\mathbf{E}\times\left(\frac{\mathbf{V}_{\mathrm{B}}}{c}\times\mathbf{E}\right) + \left(\frac{\mathbf{V}_{\mathrm{B}}}{c}\times\mathbf{B}\right)\times\mathbf{B} \\ &= -\left(E^2 + B^2\right)\frac{\mathbf{V}_{\mathrm{B}}}{c} = -\lambda\left(E^2 + B^2\right)\frac{\mathbf{V}_{\mathrm{A}}}{c} \\ &= -\lambda\left(E^2 + B^2\right)\frac{2\,(\mathbf{E}\times\mathbf{B})}{(E^2 + B^2)} = -2\lambda\,(\mathbf{E}\times\mathbf{B}) \end{aligned}$$

and, again omitting zero terms,

$$\begin{aligned} \mathbf{g} &= -\left(\frac{\mathbf{V}_{\mathrm{B}}}{c}\times\mathbf{B}\right)\times\left(\frac{\mathbf{V}_{\mathrm{B}}}{c}\times\mathbf{E}\right) \\ &= -\frac{\mathbf{V}_{\mathrm{B}}}{c}\left\{\left(\frac{\mathbf{V}_{\mathrm{B}}}{c}\times\mathbf{B}\right)\cdot\mathbf{E}\right\} = \frac{\mathbf{V}_{\mathrm{B}}}{c}\left\{\frac{\mathbf{V}_{\mathrm{B}}}{c}\cdot(\mathbf{E}\times\mathbf{B})\right\} \\ &= \lambda^2\left\{\frac{2\,(\mathbf{E}\times\mathbf{B})}{(E^2 + B^2)}\right\}\left\{\frac{\mathbf{V}_{\mathrm{A}}}{c}\cdot\left(\frac{E^2 + B^2}{2}\right)\frac{\mathbf{V}_{\mathrm{A}}}{c}\right\} \\ &= \lambda^2\left(\frac{\mathbf{V}_{\mathrm{A}}}{c}\cdot\frac{\mathbf{V}_{\mathrm{A}}}{c}\right)(\mathbf{E}\times\mathbf{B}) = \lambda^2\left(\frac{V_{\mathrm{A}}}{c}\right)^2(\mathbf{E}\times\mathbf{B}) \end{aligned}$$

Collect terms and factor out $(\mathbf{E}\times\mathbf{B})$ to get

$$\mathbf{S}' = c\,(\mathbf{E}'\times\mathbf{B}') \doteq \gamma_{\mathrm{B}}^2 c\,(\mathbf{E}\times\mathbf{B})\left\{\left(\frac{V_{\mathrm{A}}}{c}\right)^2\lambda^2 - 2\lambda + 1\right\}$$

which is eq.(15).

To derive eq.(27) we have eq.(2), eq.(14), and $(\mathbf{V}\cdot\mathbf{E}) = (\mathbf{V}\cdot\mathbf{B}) = 0$. Using eq.(14), and property (b) of footnote 13,

$$\begin{aligned} E'^2 &= \gamma_{\mathrm{B}}^2\left(\mathbf{E} + \frac{\mathbf{V}_{\mathrm{B}}}{c}\times\mathbf{B}\right)\cdot\left(\mathbf{E} + \frac{\mathbf{V}_{\mathrm{B}}}{c}\times\mathbf{B}\right) \\ &= \gamma_{\mathrm{B}}^2\left\{E^2 + 2\mathbf{E}\cdot\left(\frac{\mathbf{V}_{\mathrm{B}}}{c}\times\mathbf{B}\right) + \left(\frac{\mathbf{V}_{\mathrm{B}}}{c}\times\mathbf{B}\right)\cdot\left(\frac{\mathbf{V}_{\mathrm{B}}}{c}\times\mathbf{B}\right)\right\} \end{aligned}$$

Omitting zero terms,

$$2\mathbf{E}\cdot\left(\frac{\mathbf{V}_{\mathrm{B}}}{c}\times\mathbf{B}\right) = -2\frac{\mathbf{V}_{\mathrm{B}}}{c}\cdot(\mathbf{E}\times\mathbf{B}) \qquad \text{and}$$

$$\left(\frac{\mathbf{V}_B}{c}\times\mathbf{B}\right)\cdot\left(\frac{\mathbf{V}_B}{c}\times\mathbf{B}\right)=\left(\frac{\mathbf{V}_B}{c}\right)\cdot\left\{\mathbf{B}\times\left(\frac{\mathbf{V}_B}{c}\times\mathbf{B}\right)\right\}=\left(\frac{V_B}{c}\right)^2 B^2$$

Thus

$$E'^2=\gamma_B^2\left\{E^2-2\frac{\mathbf{V}_B}{c}\cdot(\mathbf{E}\times\mathbf{B})+\left(\frac{V_B}{c}\right)^2 B^2\right\}$$

Similarly,

$$B'^2=\gamma_B^2\left\{B^2-2\frac{\mathbf{V}_B}{c}\cdot(\mathbf{E}\times\mathbf{B})+\left(\frac{V_B}{c}\right)^2 E^2\right\}$$

Combining, and using $(\mathbf{E}\times\mathbf{B})=\left\{2\mathcal{E}/\left[1+(V_B/c)^2\right]\right\}\frac{\mathbf{V}_B}{c}$ from eqs.(2, 13, and 21), where $\mathcal{E}=\left(E^2+B^2\right)/2$, gives

$$\begin{aligned}\mathcal{E}' &= \frac{1}{2}\left(E'^2+B'^2\right)\\ &= \gamma_B^2\left\{\left[1+(V_B/c)^2\right]\frac{E^2+B^2}{2}-2\frac{\mathbf{V}_B}{c}\cdot(\mathbf{E}\times\mathbf{B})\right\}\\ &= \frac{\gamma_B^2\mathcal{E}}{1+(V_B/c)^2}\left\{\left[1+(V_B/c)^2\right]^2-4\,(V_B/c)^2\right\}\\ &= \frac{\gamma_B^2\mathcal{E}}{1+(V_B/c)^2}\left[1-(V_B/c)^2\right]^2=\mathcal{E}\,\frac{1-(V_B/c)^2}{1+(V_B/c)^2}\end{aligned}$$

which is eq.(27).

Appendix V: Detail of the Comoving System

The comoving system is defined by $\mathbf{S}'=c\,(\mathbf{E}'\times\mathbf{B}')=0$. Thus $|\mathbf{E}'\times\mathbf{B}'|=E'B'\sin\theta'=0$ where θ' is the angle between $\mathbf{E}'$ and $\mathbf{B}'$ in the comoving system.

From eqs.(7.62 and 7.63) of Rindler [14], $(E'^2-B'^2)=(E^2-B^2)$ and $(\mathbf{E}'\cdot\mathbf{B}')=(\mathbf{E}\cdot\mathbf{B})$. It follows that:

(a) An event with $(\mathbf{E}\cdot\mathbf{B})\neq 0$ has $E'B'\neq 0$ and therefore $\mathbf{E}'$ and $\mathbf{B}'$ must be either parallel or anti-parallel, $\theta'=0$ or $\theta'=\pi$ at this event;

(b) An event with $0=(\mathbf{E}\cdot\mathbf{B})=(\mathbf{E}'\cdot\mathbf{B}')=E'B'\cos\theta'$ cannot have $E'B'\neq 0$ in the comoving system because that would require both $\cos\theta'=0$ and $\sin\theta'=0$. Thus $E'B'=0$ and one of E' and B' must be zero. If $E>B$ then $E'>B'$ and hence $B'=0$. If $E<B$ then $E'<B'$ and hence $E'=0$;

(c) If both $0=(\mathbf{E}\cdot\mathbf{B})$ and $E=B\neq 0$ at an event, then both $E'B'=0$ and $E'=B'$, and therefore $E'=B'=0$ and the fields and energy density $\mathcal{E}'$ in the comoving system are zero. But eq.(2) and eq.(19) show that such an event also has $(V_A/c)=1$ and hence $(V_B/c)=1$ which is an unphysical value for a Lorentz boost velocity. The case $E=B\neq 0$ and $0=\mathbf{E}\cdot\mathbf{B}$ therefore must be approached as a limit.

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