Chaos Regained

On the Possibility of a New Era of Orbital Dynamics

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Abstract

In this paper I explore how the nature, scope, and limits of the knowledge obtained in orbital dynamics—the science concerned with the motions of bodies in the solar system—has changed in recent years. Innovations in the design of spacecraft trajectories, as well as in astronomy, have led to a new hybrid of theory and experiment, and suggest that the kind of knowledge achieved in orbital dynamics today is dramatically different from the knowledge achieved prior to those innovations. Thus, orbital dynamics may have entered a new era.

0. Introduction[1]

The Hiten spacecraft was launched by Japan in 1990 to test technologies for future lunar and planetary missions.[2] A communications failure led to the adoption of a new trajectory that eventually took Hiten into orbit around the Moon. The computed trajectory, which was designed by Belbruno and Miller (1993), required very little fuel to navigate because it exploited chaotic regions of the gravitational field produced by the Moon, the Earth, and the Sun. It represented an approximate solution to the dynamics generated by two three-body problems: the Earth-Moon-spacecraft system, and the Earth-Sun-spacecraft system.[3] Hiten was eventually deposited on a lunar weak stability boundary (WSB): a region of the gravitational field surrounding the Moon where the forces (from the Earth, Moon, and Sun) roughly cancel out.[4]

According to Belbruno, Hiten's successful navigation of the low-energy transfer “represents a paradigm shift in our perceptions of how space travel can be done” (2007, p. 138). In this paper, I point out some ways in which it may well represent something more. The calculation of the low-energy trajectory, and its successful application to the engineering problem at hand, embodies an approach to orbital dynamics[5] that is quite different from the approach taken in the three centuries following Newton's *Principia*. The difference is characterized, in large part, by the shift from focusing on forces —that act between pairs of bodies—to focusing on distinctive regions of the gravitational field through

1 Because there are so many footnotes in this paper, I think it best to point out that they are used principally to explain various technical terms—and elaborate on various technical concepts—that may be unfamiliar to the general reader.

2 Reports on what actually happened in the mission paint rather different pictures. It is possible that political concerns are at least partially to blame: compare the mission summary in (Uesugi, 1996) to that in (Belbruno, 2007). My account of the Hiten mission comes principally from the latter, where Belbruno also explains some of the events that may have led to politically sensitive reactions to Hiten's accomplishments.

3 Solutions to these two restricted versions of the three-body problem were patched together to achieve an approximate solution to what is called the planar (pseudo-circular) restricted four-body problem in three dimensions (Belbruno, 2004, p. 151).

4 There are many definitions of WSBs, and they can be quite mathematically technical. A more intuitive description is provided in (Yagasaki, p. 198): a lunar WSB “determines a transition region between the gravitational capture and escape from the Moon in the phase space”.

5 Orbital dynamics is the term I will use for that area of gravity research which includes research on spacecraft trajectory design as well as astronomy. It is concerned with the motions of bodies in the solar system.

which a single body can move (most notably, regions of high gradient). The difference is also characterized by the discovery, of theoretically chaotic dynamical systems, that in some ways motivated that shift, and the attendant concern with developing new techniques for analyzing gravitational fields and the kinds of motions to which they give rise.

The structure of this paper is as follows. In Section 1, I give a preliminary account of an 'era' of orbital astronomy, I explain what characterized the Newtonian era, and I summarize certain key differences between the characteristic features of that era and orbital dynamics today. In Section 2, I review Henri Poincaré's discovery of mathematical chaos and the empirical problems it raised. Finally, in Section 3, I discuss two new methodologies for conducting empirical research in orbital dynamics: one used in astronomy, and one that is currently emerging in the field of spacecraft trajectory design.

As a result of these innovations, the kind of knowledge of orbital motion that we can achieve has changed. The revision to the nature, scope, and limits of that knowledge, the revision to the theoretical ideas employed, and the revision to the kind of evidence gathered, suggest that we may be in a new era of orbital dynamics.

1. On the Historical Eras of Orbital Astronomy, and On the Possibility of a New Era

The history of orbital astronomy divides reasonably cleanly into three distinct eras.[6] The *Ancient era* came before the publication of Ptolemy's *Almagest*. It was characterized by piecemeal qualitative and semi-quantitative accounts of the most salient features of celestial motion: retrograde loops, eclipses, conjunctions, and so on. The Babylonians and the ancient Greeks produced these accounts by working from observations conducted over centuries. When Ptolemy introduced his groundbreaking treatise around 150 CE, he effected a transition into a new era—call it the *Ptolemaic era*—of orbital astronomy. His principal orbital model reduced the observed motions to five

6 My notion of era derives from remarks made by Neugebauer in *The Exact Sciences of Antiquity* (1969, p. 3) and *A History of Ancient Mathematical Astronomy* (1975, p. 2).

parameters,[7] and accounted for both the salient features of celestial motion and quantitative variations of those features. The account was continually revisited and reformulated over the next millennium by Islamic astronomers and philosophers, but its characterizing features—reducing complicated, irregular motions to just a handful of parameters, and using observations to construct cinematic models—remained the same.[8]

Starting in the mid sixteenth century, orbital astronomy underwent a series of empirical and theoretical reforms which ultimately led to a third era – call it the *Newtonian era*. During the transition—from the publication of *De Revolutionibus* in 1543 to the late seventeenth century—six different, comparably accurate methods of calculating planetary orbits were developed (Smith 2014, 268). Because it improved the agreement between calculations and observations by more than an order of magnitude, Kepler's method is now the best known. But by Newton's time there were five[9] other comparably accurate methods (Smith, 2014, p. 268), and it was an open question which, if any, reflected the physical processes that give rise to the observed motions, and which were mere curve-fits.

According to Neugebauer, the break with the Ptolemaic era occurred when Newton introduced dynamical methods into astronomy (1969, p. 3) in an effort to uncover the physics underlying the observed motions. By working from forces derived from observations rather than from observations directly, Newton laid the groundwork for a methodology by which to identify the physical features of the solar system that make a difference to the motions and the differences they make.[10] The

7 The five orbital parameters that Ptolemy used were: radius ratio (ratio of deferent and epicycle radii), orientation of the line of apsides, eccentricity, tropical period, and synodic period. Ptloemy also used two additional parameters to calculate planetary motion: one indicating the location of the planet on its epicycle, and one indicating the location of the epicycle's center on the deferent, for some epochal time (Evans, 1998, p. 368). As a correction, Copernicus replaced tropical period with sidereal period.

8 Perhaps the most notable departure from Ptolemy in this era is the model developed by Ibn al-Shāṭir in the 14th century. Though geocentric, it contains many of the innovations normally attributed to Copernicus (Swerdlow, p. 469): using minor epicycles to replace the equants in the orbits of Venus, Mars, Jupiter, and Saturn; giving a new theory of the Moon which eliminated an empirically obvious shortcoming of Ptolemy's lunar theory; giving a new theory of Mercury that reproduces the elliptical nature of its orbit (Fernini, 1998).

9 With Horrock's method being a refinement of Kepler's. Only the other six of the seven methods mentioned in (Smith 2014, p. 268) were used to produce tables.

10 The phrasing "features...that make a difference to the motions, and the differences they make" is taken from (Smith,

methodology worked as follows. First, the predominant forces acting on orbiting bodies are derived from the observed phenomena (Newton, 1687/1999, p. 796). Then idealized theories of the motion are constructed by calculating the dynamics that the known forces generate.[11] The idealized calculations are compared with observations, and discrepancies are used to identify additional relevant forces (which are no longer required to be predominant): that is, forces that make a difference to the motions, but that the idealized theory had left out. In other words, on this methodology, *discrepancies between theory and observation are taken to be physically informative*.[12] The additional forces are then incorporated into the idealized theory and the process is repeated, resulting in ever tighter agreement between theory and observation (Smith, 2014, p. 306).

Once the forces are discovered (using this methodology), and once the initial conditions are known to reasonably high accuracy, it is possible to calculate the subsequent motions to reasonably high agreement with observations. Let us consider the Moon's orbit as an example. The predominant gravitational force acting on the Moon is due to the Earth. If that were the *only* force acting on the Moon, then its orbit would be a Keplerian ellipse.[13] This constitutes an idealized theory of its motion. But as comparisons with observations show, the Moon does *not* follow such an ellipse. What could be giving rise to that discrepancy between the idealized, Keplerian lunar theory and observations? The obvious candidate is the Sun: the gravitational force of the Sun causes the Moon to deviate from Keplerian motion. A new idealized theory may be created by incorporating that solar force into the old idealization. Calculations based on the new theory agree with observations better than calculations based on the old theory. Armed with this new, improved theory, the process may be repeated: the new theory may be used to calculate the Moon's motion, those calculations may be compared with

2014), who adapted it from (Hart & Honoré, 1985).

11 An idealized theory is a theory that *would* hold exactly under specifiable circumstances, e.g., if all the forces have been accounted for.

12 Discrepancies must have clear signatures to be physically informative, of course (noise in the data isn't informative).

13 In this example, I ignore various complications that arise due to the fact that the Earth is oblate rather than spherical.

observations, and physical sources for any residual discrepancies—once discovered—may be used to refine the theory further. In point of fact, this is how successively more accurate lunar theories were constructed in the centuries following Newton's *Principia*.

The methodology is extremely powerful. Each time a new source for a discrepancy is discovered, it provides evidence for the physical reality not just of that source but of all the previously identified sources for discrepancies, because those other sources were presupposed in the calculation that revealed the new discrepancy in the first place. New idealized theories are, for the most part,[14] created under the constraint of retaining all previously identified sources. That is, the creation of new idealized theories tends to be a monotonic process: in the case of the Moon, for instance, the second idealization retained the previously identified gravitational force due to the Earth. Consequently, before new discrepancies can even be revealed, physical sources for other, larger discrepancies must be identified and included in the calculations (a process which reduces the overall discrepancy between calculation and observation).[15] Physical sources for somewhat smaller discrepancies are identified next, and then physical sources for still smaller discrepancies, and so on. The effect of Mars on the Moon's orbit, for instance, is detectable only once the effect of the Sun on the Moon's orbit is taken into account: the Sun's gravitational pull is so massive that it masks Mars' pull.

Because the evidence for physical sources compounds in this way, with later discoveries providing evidence for earlier ones, the methodology protects against performing orbital calculations that are mere curve-fits. If the theory was a purely mathematical approximation of celestial motion, one that did not correspond to anything physically real, then eventually a discrepancy would be uncovered

14 Sometimes, previously identified physical sources are found to produce forces that differ slightly from the forces used in the old idealized theory. As a result, the forces used in the old theory may occasionally be adjusted (and so are not quite retained in the transition to the new theory). The oblate, non-spherical shape of the Earth, for instance, produces a force which affects the Moon's motion in slightly different ways from the force that the Earth *would* produce if it were spherical. Similarly, the values of certain parameters used to calculate the forces—like the masses of the planets, say—are sometimes updated in the transition to the next idealized theory.

15 Because of this, there is a 'preferred order' in which discrepancies must be discovered: larger ones must be found first (and sources for them found first) before smaller ones become detectable. See the discussion of the Great Empirical Term in (Smith, 2014, pp. 298-306) for an example.

that had no physical source. This would imply that the underlying theory (Newtonian gravitation theory) is either wrong, or in some sense only approximately correct.[16]

These three eras,[17] which I take to be exemplars of what an 'era' is, are characterized by the distinct research methodologies employed in them and the kinds of knowledge that those methodologies achieved. The knowledge achieved in the Newtonian era (on which I focus, in order to compare it with orbital dynamics today) was largely of the forces that give rise to the observed dynamics, and the characteristic research methodology was the methodology just described.[18] To put it aphoristically: the Newtonian era was characterized by a particular combination of theory and experiment, its own *theory-experiment hybrid*. Distinct eras feature distinct hybrids. To ask whether we are in a new era is to ask whether today's hybrids, today's methodologies for investigating orbital

16 One of the most famous such discrepancies was a 43 arc-second discrepancy in the centurial precession of Mercury's perihelion, discovered in the second half of the nineteenth century (Newcomb, 1882). No physical source for the discrepancy could be found that was consistent with Newtonian theory. But Einstein's theory of gravity was able to account for that discrepancy: it is caused by time dilation, non-linearities in the spacetime metric, and the curvature of space that the Sun's mass induces. The success of Einstein's theory, with regards to the 43 arc-second discrepancy, actually served to demonstrate another deeply desirable aspect of the Newtonian-era methodology: the features of the solar system that make a difference to the motions, which the Newtonian-era methodology was used to identify, *remained the same even across theory-change*. The details of that demonstration are as follows. The observed rate of advance of Mercury's perihelion, as reported in Newcomb's tables, was roughly 5,600 arc-seconds per century. 5,025 arc-seconds came from the 26,000 year wobble of the Earth that produces the precession of the equinoxes, and 532 arc-seconds came from perturbations due to the other planets. Both the 5,025 arc-second figure and the 532 arc-second figure were *Newtonian*, however, in that calculations of them presupposed Newtonian gravitational forces. It follows that the 43 arc-second figure (which comes from subtracting 5025+532 arc-seconds from 5600 arc-seconds) also presupposed Newtonian gravity. How, then, could it possibly be used as evidence for an alternative theory of gravity like Einstein's? It could be evidence for an alternative theory so long as in that alternative, calculations of the portion of the precession arising from the Earth's wobble, and calculations of the portion of the precession arising from perturbations due to the other planets, agreed with the Newtonian-theoretic calculations *to a sufficiently high degree*. Einstein ensured that this would be the case by requiring that Newtonian gravity hold in the static, weak-field limit of his general relativistic theory. Thus, the physical sources identified by the methodology of the Newtonian era are so robust that even across a drastic change in underlying physical theory—one of the most prominent changes in physical theory in the twentieth century—the details that had been identified as making a difference to Mercury's orbital motion remained the same.

17 It would be anachronistic to refer to the Ptolemaic era (or the Ancient era) as an era of orbital 'dynamics', when the word 'dynamics' is decidedly post-Newtonian. That is why I have referred to the three eras collectively as eras of orbital astronomy. Because the contemporary study of orbital motion includes the field of space mission design as well as astronomy, I will use the phase 'orbital dynamics' to refer to the scientific study of *all* orbital motions—meaning the motions of both artificial and natural bodies—in the solar system. I will also refer to the Newtonian era as an era of orbital dynamics; as pointed out earlier, Neugebauer notes that the word 'dynamics' is not so anachronistic when applied to the Newtonian era.

18 Examples of the methodology's success include: Clairaut's derivation of the motion of the lunar apse in 1749; Laplace's analysis of the Great Inequality in the motions of Jupiter and Saturn in 1786; the hypothesized existence of a fourth outer planet to explain a 2' anomaly in the motion of Uranus (which was subsequently confirmed when Neptune was discovered) in 1846.

motions, are sufficiently distinct from the characteristic hybrid of the Newtonian era.

I do not think that a definitive answer can be given at this time; the contemporary hybrids are too nascent. But it is still worth pointing out why one might think the hybrids are sufficiently distinct, that is, why one might think that we are in a new era. By examining the distinctions closely, we will get a better sense for the specific epistemological claims of different modes of scientific inquiry, and we will also see how the discovery of theoretical and empirical problems can lead to a change in the nature, scope, and limits of the knowledge achieved by a particular science.

By way of preparation, here is a quick explanation of why contemporary orbital dynamics has departed from the traditional theory-experiment hybrid of the Newtonian era. The departure is best understood through the lens of mathematical chaos. The colloquial explanation of chaotic dynamical systems is that they produce deterministic randomness: it is impossible to predict how a trajectory in such a system will evolve. Here is a somewhat more precise characterization: the *exact* trajectory of a body under the influence of chaotic dynamics is *infinitely sensitive*[19] to initial conditions. The smallest of perturbations to those initial conditions will result in rapidly growing changes in the body's long-term behavior. In many cases, those changes are so severe that it is impossible to predict even the most general, global, qualitative characteristics of the trajectory. It is impossible to predict the 'type' of trajectory that will result from an imperceptible perturbation.

Chaos will be discussed in more detail in Section 2. For now, what is important to realize is that the discovery of chaos eventually exposed the following shortcoming in the Newtonian-era methodology: because the initial conditions of a body can never be known exactly (measurement, after all, is a finitistic process), there will always be irreducible discrepancies between idealized calculations and observations of that body's motion when the dynamics in question are chaotic. According to the

19 The phrase *infinitely sensitive to initial conditions* refers to both the sensitive dependence on initial conditions that is characteristic of chaotic dynamical systems and to the 'topologically mixed' feature of such systems; see (Hasselblatt & Katok, 2003, p. 205). Together, these two features of chaos produce the stochastic behavior which is identified with it.

methodology of the Newtonian era, this would suggest that something is amiss with Newtonian theory.[20] But that's wrong; in many systems, chaotic behavior is a perfectly natural consequence of Newtonian gravitation. So the Newtonian-era methodology fails in the case of chaotic dynamical systems: *discrepancies between theory and observation do not always correspond to some unidentified physical force affecting the trajectory*. New methodologies have since been developed to replace it: one in astronomy, and one in the field of spacecraft trajectory design.

The difference between contemporary and Newtonian orbital dynamics boils down to a difference in the *way* dynamics is studied, and in a sense, in the *kinds* of dynamical behavior studied. A key presumption of the Newtonian-era methodology was that forces are sufficient for getting at the motions of individual bodies. That is, if the initial conditions of a body are known to a reasonably accurate degree, and if the relevant forces on that body are known, then the trajectory of that body may be computed to very high accuracy. And correlatively, whenever a systematic discrepancy between calculation and observation emerged, the source had to be some shortcoming in the calculation (like insufficient convergence of an infinite series approximation) or a further force acting on the body that the calculation had left out.[21] But forces are not always sufficient for accurately calculating the motions of bodies. In the case of chaotic systems, residual discrepancies between calculation and observation will still exist *even if all the relevant forces acting on a body have been accounted for*. The forces do

20 Or rather, that something is amiss with what is called *post-parametrized Newtonian formalism*, which is a modification of Newton's equations of motion that includes corrections for relativistic effects (Will & Nordtvedt, 1972). Though there is not enough space to get into the details here, Einstein's discovery of general relativity did not inaugurate a new era in the sense of 'era' described earlier. The exact same methodology continued to work for studying the dynamics of the solar system, and the knowledge of orbital motion achieved in the Newtonian era was in most cases unaffected (or at least, not affected very much).

21 In addition to unidentified forces, there are two other possible sources of systematic discrepancies: calculations of the infinite series solution to the equations of motion have been truncated too early (that is, the approximation of the infinite series has not been carried far enough), or the fundamental constants (such as the masses) are not quite right. George Hill raises precisely the former worry in the introduction to his article on the motion of the lunar perigee: after pointing out a discrepancy between the calculated value for the mean motion of the perigee and a value derived from observations, he writes that it is unclear "whether the discrepancy should be attributed to the fault of not having carried the approximation far enough, or is indicative of forces acting on the [M]oon which have not yet been considered" (1877/1905, p. 243). It is worth noting, however, that neither of these two alternative sources tend to produce as precise a signature in the calculation-observation discrepancies that unidentified forces produce.

not tell the full story of the body's motion, when the dynamics are chaotic; the precise initial conditions play a crucial part.

For this reason, a more 'localized' approach to analyzing motion is preferable, one that focuses on the dynamics of the space surrounding the body itself. Hence the utility of the gravitational field, which characterizes gravitational tendencies at particular points. Regions of the field often exhibit an extremely high degree of variability, so that the resulting motion is extremely sensitive to location. In such regions, a body's exact initial conditions make all the difference to its long-term dynamics. Thus, chaos can be thought of as a feature of the gravity field in a particular region.[22]

In addition to chaotic regions, there are other features of the gravitational field that play an important role in the overall dynamics of the solar system. Weak stability boundaries (WSBs)—such as the lunar WSB that the Hiten trajectory exploited—constitute one class of examples. Their role in solar system dynamics is actively investigated today. Lagrange points are a more familiar example. Corresponding to every orbit are five points, L_1 through L_5, which represent locations in the gravitational field where the changes of motion produced by the forces exactly cancel. Objects can become trapped at these points, or deflected, or diverted from their original trajectories.[23]

Though Lagrange points were discovered[24] by balancing forces, the specific dynamical properties of such points are not properties of particular bodies; Lagrange points are often unoccupied. There are often no interactive forces acting on bodies at Lagrange points that could be calculated via the Newtonian-era methodology. So it is better to understand the dynamical properties associated with Lagrange points as properties of the gravitational field. Conclusions about the dynamics of bodies

22 More precisely: chaos can be thought of as a feature of the *dynamics* to which that region of the field gives rise. For the sake of concision, I will often revert to describing chaos as a feature of the field (rather than a feature of the dynamics generated).

23 L_1 has perhaps the most conceptually intuitive definition: in the case of the Jupiter-Sun system, for instance, L_1 is the point located on the line connecting Jupiter and the Sun where the gravitational force exerted by Jupiter is equal (in magnitude) and opposite (in direction) to the force exerted by the Sun. It is the point where the two forces exactly cancel.

24 Three of the Lagrange points were discovered by Euler in 1767, and the remaining two were discovered by Lagrange in 1772.

passing by, around, or through Lagrange points can then be read directly off the field itself.

The situation is similar when it comes to chaotic regions, though in such cases one *must* read the relevant dynamical properties off the field. Those properties simply aren't disclosed by the forces alone, since the range of possible dynamical behaviors is wider than the range that analyses of forces can produce. So it is not so much that the subject has changed, that astronomers were once interested exclusively in forces and that now they no longer are. Rather, it was recognized that the subject is something more general, namely, the *dynamics* of bodies under the influence of gravitational forces. And *that* subject is better tackled by analyses of the gravitational field, because the precise dynamics that the field generates cannot always be recovered from the forces acting on an individual body.

2. Poincaré and the Implications of Mathematical Chaos

Though the focus in this section is on Poincaré, many of the new techniques employed in contemporary orbital dynamics can be traced to George Hill's use of the planar circular restricted three-body problem (PCR3BP) to calculate the motion of the Moon (1877/1905; 1878/1905). In the course of his investigations, Hill found a new family of periodic solutions in the PCR3BP known as *variation curves*: orbits produced by the difference between the Sun-induced accelerations of the Moon and the Earth.[25]

Variation curves bring out a distinction between forces and fields that is worth making explicit, for it will crop up several times throughout this paper. First of all, variation curves have little to do with forces acting *on particular bodies*; of the infinitely many variation curves that go around the Earth, only the Moon's is produced by a body experiencing a force. The others—and, I would suggest, the Moon's too—are better understood as dynamical properties of the *space* surrounding the Earth. For that

25 The variation curve four our Moon depends on the number of lunations per year. Since the Moon has roughly 12.5 lunations/year, its variation curve is ovular. See (Wilson, 2010, pp. 23-25) for a summary of the history of the variation curve. See (Hill, 1878/1905, p. 335) for pictures of several different variation curves, including one that isn't ovular at all. See (Barrow-Green, 1991, pp. 24-25) for corrections that Poincaré made to Hill's calculations.

reason, variation curves are best thought of as features of the gravitational field produced by the Earth and the Sun.

In a general sense, of course, forces and fields are interdefinable. But there is an important difference between describing a dynamical system in terms of forces, that act between *pairs* of bodies, and describing that same system in terms of the dynamical space generated by the gravitational fields through which *lone* bodies move. For one, there is a very natural sense in which chaos is an attribute of regions of gravitational fields. This is discussed in Section 3.

But also, certain trajectories are most naturally understood as resulting from the dynamics of fields, not forces. Consider the variation curve that the Moon follows in Hill's idealized PCR3BP, that dynamical property of the region of space near the Earth. It changes its orientation as the Earth orbits the Sun, preserving the rotational symmetry of the Earth-Moon-Sun system. It does so smoothly, in that it remains the same for each revolution of the Moon. It slowly rotates in the symmetric space of Hill's idealized model, even though there are no external perturbations.

If we were to analyze this three-body problem in the more traditional way, looking at compounds of pairs of forces between bodies, we would never derive the variation curve. To see why, start with the interactive force between just the Earth and the Moon, which gives rise to the lunar Keplerian ellipse. That ellipse, unlike the lunar variation curve, does not change its orientation as the Earth orbits the Sun. For at this stage in the construction of mathematical models,[26] we leave out the Sun's perturbative influence. Nothing causes the Keplerian ellipse to precess, or fluctuate, or change orientation. When the Sun's perturbative force on the Moon is added into the model, the Moon's Keplerian ellipse does change its orientation by precessing. But it doesn't change its orientation in the

26 Here as elsewhere, I will use the phrase 'mathematical model' to mean 'tractable simplification' or (equivalently) 'tractable idealized theory', as is often standard. Note that Newton's full theory of gravity is not a mathematical model in this sense, for the full theory is too intractable to use for calculating the actual motions. In order to perform any calculations at all, at least some simplifications must be introduced.

smooth manner of Hill's variation curve, for the Moon never traces out the 'same' Keplerian ellipse.[27] It follows a different ellipse, with different orbital parameters, on each pass. So by analyzing orbital motions in terms of gravitational forces acting between pairs of bodies, rather than overlapping gravitational fields generated by single bodies, the Newtonian-era methodology fails to converge on elegant trajectories like the variation curve.[28]

The three-body problem is famously difficult to solve, so it was not at all obvious that such an intractable mathematical problem could be useful for empirical calculations. But Hill found a way to use the variation curve to calculate the Moon's motion, in part by dividing the space in which the three-body interaction unfolds into what are today called *Hill's regions*.[29] These allowed Ernest Brown[30] to construct a lunar theory of unprecedented accuracy. As qualitative characterizations of the phase space of the three-body problem, Hill's regions helped introduce qualitative (as opposed to quantitative) mathematical methods into orbital dynamics. The use of the three-body problem prompted E. T. Whittaker to write that Hill's innovations “may well be regarded as the beginning of the new era in Dynamical Astronomy” (Whittaker, 1899, p. 130).

Hill planted three seeds of a new approach to analyzing the motions of celestial bodies: the use of fields, of qualitative mathematics, and of the three-body problem. It took Poincaré, however, to nurture those seeds to fruition. The most important publication in this regard was the second version of Poincaré's prizewinning 1890 memoir, in which he analyzed the PCR3BP.[31] Of the memoir's many

27 Of course, these claims are all made under the idealizing assumption that the Moon does not exert a gravitational force on the Earth or the Sun.

28 In Propositions 26 through 29 of Book 3 of his *Principia*, Newton calculated an approximation to this variation curve. It was only an approximation, however, because he assumed that the curve was an ellipse with the Earth at its center. Hill was the first to perform fully rigorous calculation of the curve, making no unwarranted assumptions.

29 A Hill's region can be thought of as a region in the space of the PCR3BP to which the motion of the smallest of the three bodies is confined. More exactly, it is a projection onto position space of an invariant three-dimensional manifold associated with a constant Jacobi integral (Koon et al., 2000, p. 7).

30 See (Brown, 1986).

31 The first version contained an error, which was discovered only after Poincaré's memoir had been awarded the prize. It took the delicate political maneuverings of Mittag-Leffler to keep the error—and its significance—relatively hidden. In fact, the error was a portal of discovery: it was in correcting the error that Poincaré discovered mathematical chaos (Barrow-Green, 1991, pp. 67-69).

innovations, two have proved particularly important to the development of contemporary orbital dynamics: the qualitative, geometric approach to the mathematical theory of differential equations, and the discovery mathematical chaos.

For lack of space, I will not provide details on the former. Suffice to say: Poincaré's novel methods exploited the fact that, in many systems of differential equations, salient properties of the solutions can often be found even if the solutions themselves cannot. His methods were also conducive to analyses of fields: whereas analytical methods (series expansions, etc.) are suited to the study of forces, the geometrical features of phase space (the properties of manifolds, etc.) are most naturally understood as the properties of gravitational fields.[32]

Poincaré's discovery of chaos was perhaps the single most important breakthrough in the development of contemporary orbital dynamics. What he found was that if a periodic solution is unstable, then there are solution curves that take infinitely long to reach it and solution curves that take infinitely long to leave it. Perturbations can cause the surfaces formed by these curves to distort[33] and intersect transversally. It turns out that there are infinitely many points of intersection among the trajectories (today called *homoclinic trajectories*) winding onto and off of the intersection of the surfaces. The resulting structure has since been dubbed the *homoclinic tangle*, and its enormous complexity makes it difficult to depict to even a second-order approximation.[34] The tangle is composed

32 Modern dynamical systems theory—the core theorems, central ideas, and notation—derives largely from Poincaré, and his methods are still used in orbital dynamics today. Interestingly, this shift to a new kind of mathematics for doing orbital dynamics has historical precedent. In the Newtonian era, a new kind of mathematics—featuring quantitative, analytic, and symbolic methods—replaced the geometric, Eudoxian mathematics of the Ptolemaic era. The new mathematics allowed for solutions to problems that many (including Newton) had failed to solve using purely geometric methods. Two particularly prominent examples include Johann Bernoulli's solution for the motion of a projectile under the influence of Galilean gravity when resistance varies as the n^{th} power of velocity, and Clairaut's derivation of the entire 3^{o} mean motion of the lunar apse. Though there is not enough space to present the full argument here, I take this 'new mathematics' parallel—this parallel between the transition from the Ptolemaic to the Newtonian era, and the transition from the Newtonian era to contemporary orbital dynamics—to be another reason for thinking that we have entered, or are in the process of entering, a new era.

33 These geometric objects are called *asymptotic surfaces*. To calculate the equations for asymptotic surfaces, Poincaré calculated the equations for transverse intersections—today called *Poincaré sections*—of them. Poincaré sections have become central to dynamical systems theory. For instance, the intersection of certain types of asymptotic surfaces is now taken to be one of the hallmarks of chaotic dynamics in the solar system.

34 See (Diacu & Holmes, 1996, p. 41) for a rough depiction of first-order and second-order approximations to the

of *separatrices*: lines that separate the space of trajectories such that trajectories in different regions exhibit radically different asymptotic behavior.[35] In the homoclinic tangle, infinitely many separatrices are packed into a finite region of space, and as a result, it is practically impossible to predict the fate of any body as it passes through. Each is enmeshed in a web of separatrices, and so is infinitely close to a motley of other trajectories whose long-term behavior is quite different from its own. Hence the 'stochastic' nature of chaos: it appears to be a matter of random chance what a particular trajectory in the homoclinic tangle is liable to do, because it appears to be a matter of random chance exactly where in the web a particular trajectory is. The many different types of trajectories are too intermixed.

This is arguably the first description of chaos in a dynamical system (Barrow-Green, 1991, p. 119). Its discovery was unprecedented; nothing like chaotic motion had ever been studied or observed before.

In the particular context of orbital dynamics, the discovery of mathematical chaos—and the subsequent development of chaos theory carried out in Mary Cartwright's work on the Van der Pol equation, and John Smale's work on the horseshoe map—gave rise to a series of problems which the methods of the Newtonian era could not resolve. I articulated one of them in Section 1: the characteristic methodology of that era fails if the dynamical behavior of the bodies in question is chaotic. Poincaré was quick to identify a separate but related issue: prediction is impossible in chaotic systems because we only ever know the initial conditions of a system of bodies approximately, and thus we cannot be sure exactly what its long-term dynamics will be (1914/1996, p. 68). The stochastic character of chaotic systems confounds any attempt to calculate how a given trajectory will unfold. A

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homoclinic tangle.

35 To get a better sense for what a separatrix is, imagine a marble rolling down the ridge of a saddle. The marble's trajectory is sensitive to its initial location: it will role leftwards if initially placed slightly to the left of the ridge, and rightwards if initially placed slightly to the right. The ridge is analogous to a separatrix: it divides the space into two regions, and the trajectories in either region exhibit qualitatively different long-term behavior from the trajectories in the other region. Now, to get a sense for the mind-boggling complexity of the homoclinic tangle, imagine that the saddle (somehow) has infinitely many such ridges, and imagine that the ridges overlap so that the surface of the saddle exhibits an enormous amount of local variability. That is the situation of the homoclinic tangle.

third issue, raised by Pierre Duhem, involves the very nature of the knowledge which the science of orbital dynamics can achieve. Writing on chaotic phenomena discovered in 1898 by Jacques Hadamard, Duhem concluded that the question of whether certain trajectories are unbounded [36] or not "will always remain unanswered" (1914/1981, p. 141). Hadamard described such questions as being mathematically well-posed,[37] but ill-posed—or as Duhem puts it, "lacking meaning" (1914/1981, p. 142)—when viewed from the perspective of physical science. Due to the inexact, finitistic nature of measurements of initial conditions, such questions admit of multiple, dissimilar empirical answers.[38]

Weierstrass, who was on the committee that awarded Poincaré's memoir the prize, remarked that "[the publication of Poincaré's memoir] will inaugurate a new era in the history of celestial mechanics" (Diacu & Holmes, 1996, p. 44). Hermite, who was also on the committee, wrote: "Poincaré's memoir is of such rare depth and power of invention, it will certainly open a new scientific era from the point of view of analysis and its consequences for astronomy" (Barrow-Green, 1991, p. 134). Despite these accolades, however, it took about 90 years to recognize the empirical value of Poincaré's discovery of chaos. There are several possible explanations as to why,[39] but my own suspicion is that at the turn of the century, it was just not clear how to subjugate chaos in *mathematical*

36 The trajectories in question were geodesics on surfaces of negative curvature. Unbounded geodesics eventually go off to infinity, while bounded ones do not.

37 In that they admit of a single well-defined answer when the initial conditions are known with mathematical exactness.

38 One of these questions, Duhem hypothesized, was the question of the solar system's stability; he wrote that though the question may be meaningful for the mathematician, for the astronomer "the problem of the stability of the solar system should be...a question devoid of all meaning" (1914/1981, p. 142). Though there is not enough space to get into the details here, the issue of the solar system's stability has been heavily investigated in contemporary orbital dynamics. The consensus seems to be that the solar system is chaotic, but only "marginally" so (Malhotra et al., 2001, p. 12343): it is impossible to predict the locations of the major bodies extremely far into the future, but the solar system is stable in the sense that there is a 99% chance that none of the planets will suffer collision or ejection in the next 5 Gyr (Laskar, 2008; Batygin & Laughlin, 2008; Laskar & Gastineau, 2009). These latter results, regarding the likelihood (or lack thereof) of collision or ejection, have led some to say that the solar system is "marginally stable" (Laskar, 1996, p. 155). Altogether, these results have yielded scientifically meaningful answers to the question of stability, as well as refinements to the very notion of what stability is. I take such accomplishments to be another reason for thinking that we may have entered a new era of orbital dynamics.

39 World War I had a particularly disastrous effect on the state of French mathematics, since mathematicians (as with most all those who worked in academics) were not spared from being sent to the front. Also, Poincaré did not have that many students to begin with, and so it is only natural that it would take a while for all the nuances of his work to spread throughout the field of orbital dynamics. My thanks to Boris Hasselblatt for pointing out both of these possible explanations.

systems to rigorous analysis, and (for reasons highlighted by Duhem) chaotic behavior in *empirical* systems looked to pose even greater problems. It was not even clear that chaotic dynamics in the physical world *could* be studied scientifically.[40]

Nevertheless, the problems posed by mathematical chaos were apparent at least to Poincaré, and perhaps to a few others working at the cutting edge of mathematical astronomy.[41] The following three questions summarize just a few of the issues raised by the discovery of chaos, and the possibility of its presence in the solar system.

1. Is there any empirical evidence for the physical reality of chaotic dynamics?[42]
2. Is it possible to differentiate, in empirical practice, between motions that are infinitely sensitive to initial conditions, and motions that are highly sensitive—but not infinitely sensitive—to initial conditions?[43]
3. Is long-term prediction (based on mathematical calculations) even possible, if there are chaotic dynamical systems in the physical world?

With these questions in mind, let us turn to the new methodologies used in orbital dynamics

40 The initial impact of Poincaré's work on orbital mechanics was so small that one of the field's authoritative textbooks in the mid-twentieth century, *Methods of Celestial Mechanics*, mentions his name only once (Brouwer & Clemence, 1961, p. 17).

41 Arguably, our modern conception of what constitutes dynamical chaos is due more to the work of Cartwright and Smale than the work of Poincaré. See (Dyson, 2006) for a description of Cartwright's contribution in particular. I focus on Poincaré because he was the first to discover a chaotic dynamical system, and he was among the earliest to realize that chaotic dynamical systems pose problems for predicting phenomena.

42 This question, as well as the other two, concerns only the existence of chaos in the solar system. There are other domains in which such questions could be explored—for instance, Cartwright's research on the Van der Pol equation—but I do not do so here.

43 Recall that infinite sensitivity to initial conditions is the hallmark of chaos. By 'highly sensitive' motions, or 'highly sensitive' dependence on initial conditions, I mean to refer to motions that exhibit a high degree of instability—or a great deal of variation in the relevant orbital parameters—but that are not chaotic. Suppose, by way of illustration, that a region R of the gravitational field gives rise to the following dynamics: bodies located more than a couple meters from each other in R exhibit radically divergent long-term behavior, but when within a meter of one another, the bodies' long-term behavior is roughly the same. Then bodies in this region exhibit highly sensitive dependence on their initial conditions. But the sensitivity is not infinite, as it is for regions of the gravitational field that give rise to chaotic dynamics.

today. Contemporary orbital dynamics has had success in answering these questions, and that is another reason to think that we may have entered a new era.

3. Astronomy, and the Design of Spacecraft Trajectories

Prior to the introduction of chaos theory to empirical orbital dynamics, two mathematical breakthroughs helped demonstrate the general tractability of chaotic phenomena. First, there was the development of KAM theory. To put it very roughly, KAM theory showed that regular, nearly periodic, non-chaotic motion can survive small perturbations.[44] One of the implications of KAM theory is that for perturbations that are sufficiently small, chaotic behavior remains confined to small pockets of phase space. Consequently, predominantly non-chaotic systems are *robust* in the sense that they often remain predominantly non-chaotic even when perturbed. Second, it was discovered that the structures that give rise to chaos often persist after perturbation; small perturbations do not alter the qualitative characteristics of the dynamics generated.[45] So chaotic dynamical systems are *robust* as well: they tend to remain chaotic even when subjected to perturbations.

Both of these discoveries showed that chaos is like many other empirical phenomena, in that it appears and persists in some—but not all—dynamical systems. What is more, its robustness to perturbation allows it to be approximated: to determine whether a system of bodies gives rise to chaotic dynamics, it is enough to determine whether a suitably large fragment of that system (say, the system composed of only the largest of the bodies) does. Thus, chaos began to seem like the sort of thing that can, and should, be studied empirically.

44 KAM theory is a collection of theorems—developed in the 1950s and 1960s by Andrey Kolmogorov, Vladimir Arnold, and Jürgen Moser—which address the likelihood that perturbations of a completely integrable Hamiltonian system will render that system non-integrable. Non-integrable systems can, in some cases, exhibit chaotic behavior.

45 For example, the structure underlying Poincaré's homoclinic tangle is the horseshoe map, and it is *structurally stable* insofar as (sufficiently small) perturbations of it preserve the qualitative characteristics of the dynamics it generates. Chaos is, in that sense, a stable (or one may say, persistent) feature of the horseshoe map. Once again, my thanks to Boris Hasselblatt for pointing this out.

3.1 The Astronomy Methodology

Those empirical studies came quickly. In fact, the use of chaos in orbital dynamics ultimately led to a new methodology for astronomy. The methodology developed around two criteria: the *resonance overlap criterion* and the *exponential divergence criterion*. According to the resonance overlap criterion, as long as two orbital resonances are sufficiently close, motions that are under the influence of both are chaotic.[46] According to the divergence criterion, a numerically integrated trajectory is chaotic[47] if it diverges from nearby trajectories at an exponential rate. That rate of divergence is called *Lyapunov time*: it is the time it takes for two trajectories to diverge by a factor of *e*. Lyapunov time can be thought of as a *measure* of chaos: the more chaotic the system, the faster two nearby trajectories diverge, and so the shorter the Lyapunov time.

The criteria were first used by Jack Wisdom (1980)[48] while analyzing the PCR3BP. He found that the resonance overlap criterion can be used to predict which orbital motions are probably generated by chaotic dynamics, and which are not. But chaos is a mathematical phenomenon, strictly speaking. It has to do with the apparently unbounded sensitivity to initial conditions of numerically integrated trajectories[49] in mathematical systems. So Wisdom had to link the overlap criterion to the behavior of

46 Two clarifications of the notions used in these criteria are in order. First, resonances arise in orbital dynamics when two bodies exert periodic gravitational forces on each other. One of the more intuitive kinds is *mean-motion orbital resonance*, which occurs when two bodies have commensurate periods. Pluto and Neptune, for example, are in a 2:3 mean-motion orbital resonance: Pluto orbits the Sun almost exactly twice, for every three orbits of Neptune. Jupiter's trojan asteroids—located at its L_4 and L_5 Lagrange points—are in 1:1 mean-motion orbital resonance with Jupiter: they orbit the Sun once, for every one orbit of Jupiter. Second, there are many different ways to make the phrase 'sufficiently close' precise. The one due to Wisdom (1980) is that two resonances are sufficiently close if and only if their separatrices overlap.

47 A terminological point: when I say that a trajectory is chaotic, I mean that the trajectory is generated by a chaotic dynamical system. That is, motion along that trajectory is infinitely sensitive (in the sense specified earlier) to initial conditions. The phrase 'chaotic trajectory' is merely shorthand. Recall that chaos is best understood as a property of the dynamics generated by regions of a field. Strictly speaking, a lone trajectory is not really the sort of thing that can be chaotic.

48 That is, Wisdom was the first to use them extensively in astronomy. He adopted both from (Chirikov, 1979).

49 Numerical integration is one way to follow the path of a body in any system, chaotic or not. It is a way of determining the trajectory of a body by starting with its initial state and using the equations of motion to determine its subsequent states. Numerical integrations of trajectories, in contemporary astronomy, amount to mathematical *experiments* on the dynamical behavior of a body over time.

numerical integrations.[50] Hence the divergence criterion. Since only chaotic trajectories diverge at an exponential rate, the divergence criterion could be used to establish that a particular region of phase space is chaotic.[51] This is evidence that the corresponding region of the physical gravitational field does indeed generate chaotic dynamics, so long as that region lies within overlapping resonances. Wisdom found that numerical experiments agreed with the overlap criterion in the PCR3BP. Later, he showed that his results could explain the presence of the 3:1 Kirkwood gap (1982; 1983; 1985),[52] which is caused by the overlap of a 3:1 mean-motion resonance with secular resonances due to Jupiter and Saturn.

The core insight of this line of thought is that agreement between the resonance overlap criterion—used to identify dynamical systems in the *physical* world that are chaotic—and the divergence criterion—used to identify chaotic regions of *mathematical* models—gives us a way to connect the mathematical phenomenon of chaos with physical phenomena. Call this method for learning about gravitational fields the *astronomy methodology*.

The astronomy methodology—in particular, the hybrid of (resonance overlap) theory and (numerical) experiment which it features—is ubiquitous in contemporary orbital dynamics: all computationally identified cases of chaotic dynamics in the solar system are caused by overlapping resonances (Lecar et al., 2001, p. 581).[53] Moreover, the methodology makes crucial use of fields. Numerical experiments are no longer performed to identify forces acting between particular bodies;

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50 Prior to Wisdom, it was not at all clear how to distinguish quasiperiodic motion from motion of a truly chaotic kind. Numerical integrations involve only finitely many calculations, and so it can be hard to tell whether carrying out additional calculations will show that a seemingly random scattering of points on the Poincaré surface of section (that scattering is the thumbprint of chaos) actually lie on a simple curve (the thumbprint of quasiperiodic motion) (Wisdom, 1980, pp. 1128-1129). That is, a numerically integrated trajectory might *look* chaotic, but only because the numerical calculation was truncated too soon.

51 The divergence of nearby quasiperiodic trajectories is roughly linear, not exponential.

52 The 3:1 Kirkwood gap is an empty region in the asteroid belt that occurs at around 2.5 AUs from the Sun.

53 Lecar et al. (2001) provide a succinct explanation of just how impressive Wisdom's achievement was: he identified a straightforward dynamical process that "(a) could open a gap at one resonance, (b) in principle might be generalized to account for other Kirkwood gaps, and as an added bonus, (c) could deliver asteroidal fragments into the inner solar system as meteorites" (p. 588).

indeed, for the reasons given in Section 1, they *cannot* be used for that purpose if chaotic dynamics are present. Rather, numerical experiments are performed to uncover the qualitative properties of the field regions in question, and to evaluate the limits of calculations when those regions are chaotic. This is another reason for thinking that we are entering a new era.

Of course, this is not to say that numerical experiments make no use of forces. Calculations are often conducted using force equations. But the conclusions that can be drawn from calculations based on forces are inherently limited. Only by studying the properties of fields can those limitations be understood.

Lyapunov time is an extremely useful tool for understanding those limitations, in at least two respects. First of all, the Lyapunov time of a region of a field is a *theory-mediated measure* of how chaotic that region is. This is because to use it as a measure of *empirical* chaotic systems,[54] one must rely on the theory (resonance overlap theory) that links the numerical experiments to the physical world. Second, and more significantly, Lyapunov time is *meta-theoretical*: it tells us how far integrations may be carried before becoming empirically meaningless.[55] Trajectories in a chaotic dynamical system whose Lyapunov time is large can be integrated far into the future before their accuracy declines significantly.[56] Trajectories in a chaotic dynamical system whose Lyapunov time is small cannot be integrated for very long: they quickly diverge from trajectories that have empirically indistinguishable initial conditions.[57] So Lyapunov time can be used to address Poincaré's worry about

54 As opposed to using it as a measure of chaos in a mathematical model.

55 "Meaningless" in Duhem's sense. That is, Lyapunov time tells us how far integrations may be carried before they lose their claim to even approximately represent the empirical world

56 For example, the entire solar system is chaotic, and its Lyapunov time on the order of 50 Myr. So integrations can be carried millions of years into the future before losing their claim to approximately represent the world. To give just one example of this, after 10 Myr, a discrepancy of 15 m in the measurement of the initial conditions of the Earth will lead to an error of only 150 m in Earth's location. But after 100 Myr, that error grows to 150,000,000,000 m (Laskar, 2013, p. 260).

57 Kepler-36, for instance (a planetary system about 1530 lightyears from our own), is chaotic and has a Lyapunov time of roughly 10,000 days (Deck et al., 2013, p. 1). As a result, calculations of trajectories of planets in that system are only accurate in the short term; an undetectable adjustment of the initial conditions of the system, over the course of a mere 50 years, could result in a dramatically different final state.

the limits of predictability: though our knowledge of the motions is limited, we have a sound meta-theory of those limitations. We can measure our ability to predict the physical motions, and the limitations on our knowledge of them, using Lyapunov time.

Because of this meta-theoretical measure of our capacity to make meaningful predictions, the new methodology can address problems which the Newtonian-era methodology cannot. The problem of determining the precise trajectory of Halley's comet is a striking example of this. The history of attempted solutions also illustrates the problems that arise when one focuses on the forces acting between *pairs* of bodies, rather than on fields and the *lone* bodies that pass through them.

Observations of Halley's comet (also called P1/Halley), which stretch back over two millennia, occupy a special place in the history of astronomy. The problem of predicting P1/Halley's time of perihelion passage was of great importance to the acceptance of Newtonian gravitation. In the seventeenth century there was considerable debate over whether or not observations of certain comets were, in fact, observations of one and the same celestial body.[58] Newton's account of cometary motion in the *Principia*, in which he argued that they were, constituted an enormous step forward in the theoretical study of comets. Halley used it to deduce that P1/Halley orbits the Sun about once every 75 years, and that, having last reached perihelion in 1682, P1/Halley would next reach perihelion sometime between 1757 and 1759. Several decades later, Clairaut performed a massive numerical integration to make that prediction more precise; in addition to the influence of the Sun, his model also included perturbations from Jupiter and Saturn. It was generally considered a remarkable success of Newtonian gravitation theory that P1/Halley reached perihelion, in 1759, at a date that was within Clairaut's error margin (about 30 days).[59]

58 As late as 1681, Newton himself dismissed the idea that comet trajectories were periodic, that comets could 'button-hook' around the Sun (Newton, 1960/1687, pp. 340-342).

59 See (Waff & Wilson, 1995, pp. 69-86) for more details on the history of Clairaut's calculation. The calculation took Clairaut roughly six months of constant work, despite the fact that he had help from two people: the astronomer Joseph-Jérôme Lefrançais de Lalande, and Mme Nicole-Reine Étable de Labriére Lepaute. A jealous, and possible romantic interest of Clairaut's asked him not to include a written appreciation of the latter's work in his final report (Waff &

Over the centuries, many different discrepancies between calculations and observations of P1/Halley's trajectory were explained by identifying additional forces that were making a difference to its orbital motion. But predictions of the time of return were never "completely successful" (Brady & Carpenter, 1971, p. 728); there were discrepancies that could not be identified with any physical source. After reviewing several failed attempts,[60] the authors of a review article summarized the situation thus: "it appears that a large number of comets experience forces that are not included in our present gravitational model of the solar system" (Duncombe et al., 1973, p. 146).

This is exactly what scientists working with Newtonian-era methods would, and should, conclude: discrepancies between calculation and observation indicate that some relevant force remains to be identified. That force may not be gravitational; it could be the result of outgassing, for instance. Nonetheless, it is assumed that *some force acting on the comet, gravitational or otherwise, is responsible for the discrepancy*.

But as it turns out, P1/Halley's orbital parameters probably undergo chaotic variations. So it is incorrect to conclude that unidentified forces must be responsible for the discrepancy between theory and observation. Chaos is probably responsible for at least part (maybe most) of it.[61] Only partly, however, because a portion of the discrepancy may be due to physical influences on its motion (outgassing, say) that have been left out of the idealized theories.

But this raises a problem: how can additional relevant influences be identified, if P1/Halley's motion is chaotic? The irreducible discrepancies that result from chaotic dynamical systems seem like they would 'mask' the signatures of further discrepancies, blocking us from identifying further physical influences on P1/Halley's motion.

Wilson, 1995, p. 84).

60 In 1972, for instance, Brady suggested that a hypothetical tenth planet (beyond the orbit of Pluto) could be responsible for the discrepancies. He calculated what its orbit and mass would have to be, to reduce the discrepancies of P1/Halley's seven most recent apparitions at that time (from 1456 to 1910) by 93%. As Seidelmann et al. (1972) quickly showed, however, such a planet could not exist. If it did, the motions of the other outer planets would be significantly different.

61 See (Chirikov & Vecheslavov, 1989; Froeschelé & Gonczi, 1988; Muñoz-Gutiérrez et al., 2015).

But those physical influences can still be identified, so long as the corresponding discrepancies are detected on *sufficiently short time scales* – that is, short relative to the dynamical system's Lyapunov time. Over a brief enough period, there will not be enough time for discrepancies (due to the chaotic features of the gravitational field) to accumulate to any significant extent. It will be possible to find other influences on P1/Halley's motion *so long as those influences are found within that time period*. In the case of Halley's comet, the Lyapunov time looks to be roughly 70 years (Muñoz-Gutiérrez et al., 2015). Therefore, additional influences can be identified so long as their effects can be detected over times scales on the order of 70 years or so.[62] Because of this, Lyapunov time is also a measure of the temporal range over which the methodology of the Newtonian era is applicable.

Working with fields is what makes this preservation of the 'old' methods possible. If astronomers worked solely with forces acting on a given body, then the discovery of a single irreducible discrepancy would put a stop to the whole methodology. That discrepancy would dominate all others, and there would be no obvious way to study the smaller ones. But after calculating the Lyapunov times of regions of fields, astronomers can continue to use the Newtonian-era methodology so long as they adjust the time scale over which it is applied.

One might have the following concern about the conclusions, regarding whether or not a particular region of space gives rise to chaotic dynamics, that can be drawn via the astronomy methodology. No mathematical model of the solar system can account for all the gravitational forces present, of course. And recall that chaotic trajectories are infinitely sensitive to initial conditions. To name a particularly famous example, Sussman and Wisdom (1988) performed two numerical integrations of Pluto's motion. They found that if the numerical integrations are otherwise alike except for a mere *1 mm difference* in Pluto's starting position, then in 845 million years the two 'Plutos' will

62 Though the details are unimportant here, the evidential logic is occasionally a bit more complicated than this description suggests. The time scale over which influences can be identified depends on the strength of that influence. For instance, if the influence is sufficiently strong, then it can be detected over time scales longer than the Lyapunov time.

have completely different orbits.[63]

But in their calculations, Sussman and Wisdom did not take into account the gravitational forces of the asteroids. Did that affect their results? Even today, asteroids are rarely taken into consideration when constructing idealized models of orbital motion.[64] So one might be concerned that chaos is only a feature of *idealized* models. One might worry that Pluto's orbit only *seems* to be chaotic, because so many small gravitational forces (due to asteroids) were left out of the calculations. Perhaps when some of those diminutive forces are included in the model, Pluto's orbit won't seem to be chaotic after all.

The mathematical results achieved in non-linear dynamics in the middle of the twentieth century ameliorate these concerns. Chaos, recall, is *persistent*: the structures that give rise to chaos can persist through small perturbations. So though these mathematical models[65] of the solar system are incomplete, predictions made by studying them (regarding which regions are chaotic) are not especially fragile. The astronomy methodology can be used to identify chaotic regions of the gravitational field even if some gravitational forces are left out of the model.

Before discussing contemporary research on spacecraft trajectories, consider the following answers—which the astronomy methodology provides—to the questions that concluded Section 2.

1. Is there empirical evidence for chaos? Probably. Whenever resonances overlap, numerical experiments (in conjunction with the above methodology) can be used as evidence for the claim that the corresponding region of the field generates chaotic dynamics. The evidence is not strong, however, because the chaotic regions of the field have not been tested directly. It is not clear whether the

63 One Pluto could be found on one side of the Sun, and the other Pluto on the opposite side. The orbital parameters of the Plutos—the inclinations of their orbits, their orbital eccentricities, and so on—could also be radically different. Thus, an undetectable, unmeasurable change in Pluto's initial conditions leads to a totally different orbit. It is therefore impossible to predict, just a few hundred million years into the future, where Pluto will be.

64 The most up-to-date lunar ephemerides, for instance, includes perturbations due to the Sun, the planets, and 343 different asteroids (Folkner et al., 2014, p. 4). Impressive and unprecedented as that is, the ephemerides still leaves out tens of thousands of asteroids whose gravity has a small but nonzero effect on the Moon's motion.

65 Recall that the phrase 'mathematical model' means something like 'tractable simplification', as is often standard in the scientific literature. Newton's full theory of gravity, though of course a 'complete' theory of motion in the solar system, is not the sort of mathematical model that can be used to perform calculations. It is too calculationally intractable.

dynamics generated by those purportedly chaotic regions are indeed infinitely sensitive to initial conditions, or only highly sensitive. Hence the following question.

2. Is it possible to differentiate between infinite sensitivity and merely high sensitivity to initial conditions? This remains unclear. The rate of divergence of nearby trajectories can be used to distinguish quasiperiodic trajectories from chaotic trajectories in a model (by the divergence criterion). But that is still just a feature of a mathematical idealization, and it is not clear how well agreement with the overlap criterion translates that mathematical property into a property of the actual gravitational field.[66] Some sort of independent assessment is needed.

3. Is prediction even possible? Yes and no. No, because of the unpredictability of long-term dynamical behavior in chaotic systems. Yes, because we can measure the time scales over which the motions induced by chaotic dynamics are predictable, and adjust our research strategies accordingly.

Note a common theme in these answers: though a whole new methodology has been erected, it is not clear exactly how useful that methodology is for analyzing chaos. The astronomy methodology, though impressive on its own, requires independent evaluation.

3.2 The Trajectory Methodology

Recent successes in the design of spacecraft trajectories go some way towards providing that evaluation. Innovations in trajectory design have led to a second methodology for getting evidence about the dynamical properties of the solar system's gravitational field. The convergence of the two methodologies—if indeed they continue to converge—promises to yield extremely high-quality evidence for those chaotic features of the solar system that make a difference to the orbital motions. And this convergence is another reason to think that we have entered, or are in the process of entering,

66 It would be question-begging to claim that agreement between the overlap and divergence criterion is enough to differentiate infinitely sensitive and highly sensitive trajectories. The utility of that very agreement, as a means of identifying chaos, is precisely what is at issue.

a new era of orbital dynamics.

The methodology depends on using spacecraft to repeatedly approximate trajectories whose exact dynamics could only be caused by chaotic regions of the gravitational field. First, a *possibly* chaotic, low-energy trajectory is identified by using theoretical analyses that are checked against numerical computations.[67] Then, a ΔV[68] calculation is performed for the low-energy maneuvers that are needed to keep the spacecraft on that trajectory.[69] A successful navigation establishes a lower bound on how sensitive that trajectory is to the initial conditions: the larger the total ΔV expenditure, the more fuel is required to keep the spacecraft near the calculated low-energy trajectory, and so the more sensitive that trajectory must be.

Though a single successful run does not prove that the trajectory followed is infinitely sensitive to initial conditions, it is reasonably strong evidence for infinite sensitivity. The trajectory is at least thus-and-so sensitive (as measured by the total ΔV expended). And in the mathematical analysis which was conduced prior to the mission's start, that trajectory was predicted to be chaotic. These two facts suggest that the actual trajectory was indeed infinitely sensitive to initial conditions.

Successful refinements to the trajectory can actually raise that lower bound.[70] The way the evidential logic works, to raise that lower bound, is a bit intricate. It depends on how large the Lyapunov time is compared to the time scale over which the trajectory unfolds. If the time spent on the trajectory is much less than the calculated Lyapunov time, then it will be possible to eliminate *all* the ΔV expenditures required to compensate for inexact initial conditions. This shows that the trajectory is *extremely* sensitive to initial conditions, but not that it is *infinitely* sensitive. For instance, suppose a

67 This step in the methodology is very similar to the methodology used in astronomy, in that agreement between theoretical analysis and numerical calculation is taken to be evidence that the dynamics generating a particular trajectory are chaotic.

68 ΔV is a standard measure of the impulse force needed to perform a particular maneuver.

69 The ΔV expenditure required once the spacecraft is already on the trajectory, that is.

70 That is, by performing more exact calculations of the various nuances of the gravitational field (the WSBs, Lagrange points, etc.), the ΔV required to compensate for the inexact specifications of initial conditions can be reduced.

spacecraft SC follows a trajectory through a region that, according to our dynamical models, has a Lyapunov time of around 42 days.[71] If SC only follows that trajectory for 10 days, then even if no trajectory correction maneuvers[72] are performed (and so, no ΔV is expended), SC will likely end up very near to where we predicted it would. The chaotic dynamics of the region will not have enough time to make a significant difference to SC's final location.

If the time spent on the trajectory is on the order of (or longer than) the calculated Lyapunov time, however, then a certain amount of ΔV will *always* be required to compensate for the impossibility of measuring the initial conditions exactly. If SC follows its trajectory for 365 days, for instance, and no trajectory corrections are performed, then our calculations of SC's final location will almost certainly be off by many factors of *e*. To avoid such radical divergence, a number of ΔV corrections would have to be performed as SC follows its trajectory. In general, this *ineliminable need* to perform low-energy ΔV corrections—to avoid diverging from the calculated trajectory—is evidence that the calculated trajectory is chaotic.[73] Those ΔV corrections may still be reduced (though not without bound) by performing more and more accurate calculations on our mathematical models, all while keeping the spacecraft close to the purportedly chaotic trajectory. It may also be possible to continually raise the lower bound on the sensitivity of the gravitational field by performing longer and longer space missions, following trajectories that unfold on time scales greater than the Lyapunov time. If refinements like these are performed, then trajectories in the relevant region of the gravitational field have no upper bound on their sensitivity. They are infinitely sensitive, therefore chaotic.

Call this method for learning about gravitational fields the *trajectory methodology*. The novelty

71 This is, in fact, the Lyapunov time corresponding to chaotic fluctuations in the displacement of Hyperion's spin axis from its orbit normal (Wisdom et al., 1984, p. 147).

72 Trajectory correction maneuvers are rocket burns, and other ΔV expenditures, which serve to keep the spacecraft close to its calculated trajectory.

73 If no corrections are needed, but the trajectory is nevertheless chaotic, then the spacecraft's initial conditions must have been *exactly* right to follow it. But the probability of that is, mathematically, zero. So it is almost certainly the case that the calculated trajectory was never chaotic to begin with, and hence, neither was the surrounding region of phase space. Since most of those surrounding regions contain the trajectory that the spacecraft followed, it is reasonable to infer that the spacecraft's trajectory was not chaotic either.

of the trajectory methodology—its unique theory-experiment hybrid, characterized by theoretical analyses of low-energy orbits which actual spacecraft can directly test—is another reason for thinking that we are entering a new era of orbital dynamics. The kind of knowledge achieved is *direct knowledge* of the dynamics induced by the gravitational field. It is not mediated by numerical experiments, or theories of resonance overlap. We can surf the chaotic dynamics as they actually unfold in physical space.[74]

The ability to carry out experiments like these, on regions of *extremely local sensitivity* in the solar system's gravitational field, is entirely new. More traditional spacecraft missions—that do not follow low-energy trajectories—do not provide novel evidence for various fine-grained features of the gravitational field. The Trans-lunar Injections[75] used in the Apollo missions, for instance, did not confirm that the gravitational field between the Earth and the Moon gives rise to some new kind of dynamical behavior. Those injections represented patchwork solutions to familiar two-body problems: the Earth-spacecraft problem and the Moon-spacecraft problem. There was no mystery about the nature of the dynamics of those two-body problems either. The solutions to those problems are conic sections, just as with all such problems wherein a single inverse-square force acts on a single body. And there was no mystery about what forces were acting on the spacecraft: they were the same forces—one due to the Earth, the other due to the Moon—that had already been observed to affect many different celestial bodies (such as the observed effect of the Moon on the Earth, not to mention the obvious effect of the Earth on the Moon).

Missions like the Hiten's are fundamentally different. They provided evidence not of forces acting on bodies but of the dynamics generated by the fields through which bodies pass. And they

74 Fortunately, those dynamics evolve at just the right rate: slowly enough that we can analyze and manipulate them (unlike the weather), but quickly enough that we can actually observe the large-scale effects (unlike the chaotic behavior of the planets, which is far too gradual for us to observe).

75 In these maneuvers, the spacecraft is assumed to be under the influence of just one celestial body at a time. When transferring into a lunar orbit, for instance, the spacecraft is first assumed to be under the influence of just the Earth. Once it reaches a certain point in its trajectory, however, it is assumed to be under the influence of just the Moon.

provide evidence for a new kind of dynamics that can arise in three-body (not two-body) problems: extreme—even infinite—sensitivity to initial conditions. So though space travel has been around for half a century, only within the last few decades have spacecraft begun to provide evidence for various surprising features of the gravitational field.

The trajectory methodology serves as an independent test of the astronomy methodology. First, the overlap and divergence criteria may be used to determine which regions of the gravitational field are probably chaotic, and what other properties those regions exemplify.[76] Any identified chaotic dynamics could then be experimentally confirmed by testing with spacecraft. The divergence criterion could also be tested by sending two spacecraft along the 'same' trajectory simultaneously[77] and measuring the rate at which they diverge. Altogether, the convergence of the trajectory methodology with the astronomy methodology has the potential to provide us with extremely high evidence for the physical reality of chaotic features of the solar system's gravitational field.

The trajectory methodology has yet to be carried out in its entirety: no sequence of missions has produced those successive refinements to the same trajectory. And that is perhaps the most important evidential step. For if the trajectory is indeed chaotic, corrections compensate for the impossibility of ever matching the initial conditions with sufficient exactness.[78] But if the trajectory is not chaotic, then sufficiently accurate calculations will eventually eliminate the need for any corrections at all.

There is a natural way to carry out the trajectory methodology. Space agencies could begin traveling the Interplanetary Superhighway (IPS), a network of gravitational 'tunnels' that wind around the Sun, the planets, and moons, connecting their various Lagrange points.[79] The IPS, which has been described as a massive low-energy transport network (Lo & Ross, 2001, p. 8), could be navigated

76 Indeed, this is what those who design such trajectories do: the halo orbits followed by the Genesis spacecraft (whose mission lasted from 2001 to 2004), for example, were calculated using theoretical analyses of the three-body problem (and resonances therein) in conjunction with numerical computations (Howell et al., 1997).
77 That is, by making their initial conditions as near as possible.
78 Over sufficiently long periods of time, of course.
79 Belbruno and Miller relied on the dynamics of the IPS in their rescue of the Hiten.

repeatedly; thus, the trajectory methodology could be executed in full.

Let us revisit the two questions which the astronomy methodology did not definitively answer.

1. Is there empirical evidence for chaos? Yes. The success of Hiten and other missions is strong evidence for the physical reality of chaotic dynamics, especially given the agreement between the spacecrafts' actual motions and results produced by the astronomy methodology. Through successive refinements of trajectories, the possibility of empirical research on the IPS promises to strengthen that evidence considerably. Moreover, the independent confirmation of the astronomy methodology (which the trajectory methodology seems to be providing) makes it reasonable to suppose that the astronomy methodology correctly predicts which other regions of the solar system's gravitational field are chaotic. Though no spacecraft have followed trajectories originating in the 3:1 Kirkwood gap, for instance, it is reasonable to suppose that the spacecraft methodology would find that region to be chaotic too.

2. Is it possible to differentiate between infinite sensitivity and merely high sensitivity to initial conditions? Yes. That is the achievement of the trajectory methodology, in particular the step in the methodology where the trajectory is refined by carrying out additional space missions.

4. Conclusion

Two last points on the epistemology of contemporary orbital dynamics.

First, what chaos has shown is that every claim about the motions of celestial bodies must be qualified: if the dynamics giving rise to the motions are chaotic, then the claim only holds over time periods that are small compared to the Lyapunov time. Duhem was right, in a sense: claims about the precise motions are meaningless outside those bounds. Our knowledge of celestial motion is therefore limited; we cannot know a trajectory indefinitely far into the future and past. So just as Einstein circumscribed the domain to which Newtonian gravity holds to high approximation, contemporary orbital dynamics has circumscribed regions—*temporal* regions, demarcated by Lyapunov times—

within that domain (with or without general relativistic corrections) in which predictions are reasonably accurate. Second, the knowledge obtained by contemporary orbital dynamics concerns the motions of bodies, as it always has. But that knowledge is now obtained through analyses of fields, not forces acting on individual bodies. The methodologies used to learn about the solar system—and the unique hybrids of theory and experiment that they employ—are unlike those of the Newtonian era.

All in all, I find myself persuaded that we are on the cusp of a new era of orbital dynamics.

References

Barrow-Green, J. (1991). *Poincaré and the Three-Body Problem*. Providence, RI: American Mathematical Society.

Batygin, K. & Laughlin, G. (2008). On the Dynamical Stability of the Solar System. *The Astrophysical Journal, 683*, 1207-1216.

Belbruno, E. A. (2004). *Capture Dynamics and Chaotic Motions in Celestial Mechanics*. Princeton: Princeton University Press.

Belbruno, E. A. (2007). *Fly Me to the Moon*. Princeton: Princeton University Press.

Belbruno, E. A. & Miller, J. K. (1993). Sun-perturbed Earth-to-Moon transfers with ballistic capture. *Journal of Guidance, Control, and Dynamics, 16*(4), 770-775.

Brady, J. L & Carpenter, E. (1971). The Orbit of Halley's Comet and the Apparition of 1986. *The Astronomical Journal, 76*(8), 728-739.

Brown, E. W. (1896). *An Introductory Treatise on the Lunar Theory*. Cambridge, MA: Cambridge University Press.

Brouwer, D. & Clemence, G. M. (1961). *Methods of Celestial Mechanics*. New York, NY: Academic press.

Chirikov, B. V. (1979). A universal instability of many-dimensional oscillator systems. *Physical Reports, 52*, 265-376.

Chirikov, B. V. & Vecheslavov, V. V. (1989). Chaotic dynamics of comet Halley. *Astronomy and Astrophysics, 221*, 146-154.

Deck, K. M., Holman, M. J., Agol, E., Carter, J. A., Lissauer, J. J., Ragozzine, D., & Winn, J. (2012). Rapid Dynamical Chaos in an Exoplanetary System. *The Astrophysical Journal Letters, Que 755(L21)*, 1-7.

Diacu, F. & Holmes, P. (1996). *Celestial Encounters: The Origins of Chaos and Stability*. Princeton, NJ: Princeton University Press.

Duhem, P. (1981). *The Aim and Structure of Physical Theory*. (P. P. Wiener, Trans.). Forge Village, MA: Atheneum. (Original work published 1914)

Duncombe, R. L., Seidelmann, P. K., & Klepczynski, W. J. (1973). Dynamical Astronomy of the Solar System. *Annual Reviews of Astronomy and Astrophysics, 11*, 135-154.

Dyson, F. (2006). Mary Lucy Cartwright: Chaos Theory. In N. Byers & G. Williams (Eds.), *Out of the Shadows: Contributions of Twentieth-Century Women to Physics* (pp. 169-177). London: Cambridge University Press.

Evans, J. (1998). *The History and Practice of Ancient Astronomy*. New York, NY: Oxford University Press.

Fernini, I. (1998). *A Bibliography of Scholars in Medieval Islam: 150-1000 A.H*. Cultural Foundation of Abu Dhabi.

Folkner, W. M., Williams, J. G., Boggs, D. H., Park, R. S. & Kuchynar, P. (2014). The Planetary and Lunar Ephemerides DE430 and DE431. *IPN Progress Report* (pp. 42-196). Retrieved from http://ipnpr.jpl.nasa.gov/progress report/42-196/196C.pdf

Froeschlé, C. & Gonczi, R. (1988). On the Stochasticity of Halley Like Comets. *Celestial Mechanics, 43*, 325-330.

Hart, H. L. & Honoré, T. (1985). *Causation in the Law* (2nd ed.). Oxford: Oxford University Press.

Hasselblatt, B. & Katok, A. (2003). *A First Course in Dynamics*. New York, NY: Cambridge University Press.

Hill, G. (1905). On the part of the Motion of the Lunar Perigee which is a Function of the Mean Motions of the Sun and Moon. In *The Collected Mathematical Works of George William Hill* (Vol. 1). Washington: Carnegie Institute of Washington. (Original work published 1877)

Hill, G. (1905). Researches into the Lunar Theory. In *The Collected Mathematical Works of George William Hill* (Vol. 1). Washington: Carnegie Institute of Washington. (Original work published 1878)

Howell, K. C., Barden, B. T., & Lo, M. W. (1997). Application of Dynamical Systems Theory to Trajectory Design for a Libration Point Mission. *The Journal of the Astronautical Sciences, 45*, 161-178.

Koon, W., Lo, M. W., Marsden, J.E., & Ross, S. D. (2000). Constructing a Low Energy Transfer Between Jovian Moons. *Contemporary Mathematics, 292*, 129-145.

Laskar, J. (1996). Large Scale Chaos and Marginal Stability in the Solar System. *Celestial Mechanics and Dynamical Astronomy, 64*, 115-162.

Laskar, J. (2008). Chaotic diffusion in the Solar System. *Icarus, 196*, 1-15.

Laskar, J. (2013). Is the Solar System Stable? In B. Duplantier. S. Nonnenmacher, & V. Rivasseau (Eds.), *Chaos: Poincaré Seminar 2010*. Orsay, France: Birkhäuser.

Laskar, J. & Gastineau, M. (2009). Existence of collisional trajectories of Mercury, Mars and Venus with the Earth. *Nature, 459*, 817-819.

Lecar, M., Franklin, F., Holman, M., & Murray, N. (2001). Chaos in the Solar System. *Annual Reviews of Astronomy and Astrophysics, 39*, 581-631.

Lo, M. & Ross, S. (2001). *The Lunar L1 Gateway: Portal to the Stars and Beyond*. Paper presented at the AIAA Space 2001 Conference, Albuquerque, New Mexico. Retrieved from http://www.gg.caltech.edu/~mwl/publications/papers/.

Malhotra, R. & Williams, J. G. (1997). Pluto's heliocentric orbit. In S. A. Stern & D. Tholen (Eds.), *Pluto and Charon* (pp. 127-158). Tuscon, AZ: University of Arizona Press.

Muñoz-Gutiérrez, M. A., Reyes-Ruiz, M., & Pichardo, B. (2015). Chaotic Dynamics of Comet 1P/Halley; Lyapunov Exponent and Survival Time Expectancy. *Monthly Notices of the Royal Astronomical Society, 447*, 3775-3784.

Neugebauer, O. (1969). *The Exact Sciences in Antiquity* (2nd ed.). Mineola, NY: Dover.

Neugebauer, O. (1975). *A History of Ancient Mathematical Astronomy* (Vols. 1-3). Berlin: Springer-Verlag.

Newcomb, S. (1882). Discussion and Results of Observations on Transits of Mercury from 1677 to 1881. *Astronomical Papers, 1*, 363-487. U.S. Nautical Almanac Office.

Newton, I. (1960). *The Correspondence of Isaac Newton*. H. W. Turnbull (Ed.). New York: Cambridge University Press.

Newton, I. (1999). *Mathematical Principles of Natural Philosophy* (I. B. Cohen & A. Whitman, Trans.). London: University of California Press. (Original work published 1687, 1713, & 1726)

Poincaré, H. (1996). *Science and Method* (F. Maitland, Trans.). Bristol: Thoemmes Press. (Original work published 1914)

Smith, G. E. (2014). Closing the Loop: Testing Newtonian Gravity, Then and Now. In Z. Biener & E. Schliesser (Eds.), *Newton and Empiricism* (pp. 262-351). New York, NY: Oxford University Press.

Sussman, G. J. & Wisdom, J. (1988). Numerical Evidence that the Motion of Pluto is Chaotic. *Science, 241*, 433-437.

Sussman, G. J. & Wisdom, J. (1992). Chaotic Evolution of the Solar System. *Science, 257*, 56-62.

Swerdlow, N. M. (1973). The Derivation and First Draft of Copernicus's Planetary Theory: A Translation of the Commentariolus with Commentary. *Proceedings of the American Philosophical Society, 117*(6), 423-512.

Uesugi, K. (1996). Results of the Muses-A "Hiten" Mission. *Advances in Space Research, 18*(11), 69-72.

Waff, C. B. & Wilson, C. (1995). Predicting the mid-eighteenth-century return of Halley's Comet. In R. Taton & C. Wilson (Eds.), *The General History of Astronomy* (Vol. 2, pp. 69-86). Cambridge:

Cambridge University Press.

Whittaker, E. T. (1899). Report on the Progress of the Solution of the Problem of Three Bodies. *BAAS Report 1899*, 121-159.

Will, C. M. & Nordtvedt, K (1972). Conservation Laws and Preferred Frames in Relativistic Gravity. I. Preferred-Frame Theories and an Extended PPN Formalism. *The Astrophysics Journal, 177*, 757-774.

Wilson, C. (2010). *The Hill-Brown Theory of the Moon's Motion: It's Coming-to-be and Short-lived ascendancy (1877-1984)*. New York, NY: Springer.

Wisdom, J. (1980). The Resonance Overlap Criterion and the Onset of Stochastic Behavior in the Restricted Three-Body Problem. *The Astronomical Journal, 85*, 1122-1133.

Wisdom, J. (1982). The Origin of the Kirkwood Gaps: A Mapping for Asteroidal Motion near the 3/1 Commensurability. *The Astronomy Journal, 87*, 577-593.

Wisdom, J. (1983). Chaotic Behavior and the Origin of the 3:1 Kirkwood Gap. *Icarus, 56*, 51-74.

Wisdom, J. (1985). A Perturbative Treatment of Motion near the 3/1 Commensurability. *Icarus, 63*, 272-289.

Wisdom, J., Peale, S. J., & Mignard, F. (1984). The Chaotic Rotation of Hyperion. *Icarus, 58*, 137-152.

Yagasaki, K. (2004). Sun-Perturbed Earth-to-Moon Transfers with Low Energy and Moderate Flight Time. *Celestial Mechanics and Dynamical Astronomy, 90*, 197-212.